Emergence of complexity in opinion propagation: A reaction-diffusion model

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Abstract

We analyze a model designed to describe the spread and accumulation of opinions in a population. Inspired by the *social contagion* paradigm, our model is built on the classical SIR model of Kermack and McKendrick and consists in a system of reaction-diffusion equations. In the scenario we consider, individuals within the population can adopt new opinions *via* interactions with others, following some simple rules. The individuals can gradually adopt more complex opinions over time.

Our main result is the characterization of a *maximal complexity* of opinions that can persist and propagate. In addition, we show how the parameters of the model influence this maximal complexity. Notably, we show that it grows almost exponentially with the size of the population, suggesting that large communities can foster the emergence of more complex opinions.

Key words: reaction-diffusion systems, SIR models, Fisher-KPP dynamics, spreading speed, propagation properties, social contagion, propagation of opinions, emergence of complexity.

MSC2020: 35K40, 35K57, 35B40, 91D25.

1 Introduction

1.1 Motivation: the spread of ideas and opinions

The mathematical modeling of the propagation of ideas, opinions, rumors, knowledge, or other "social traits" has been envisioned since at least the 18th century [34]. The concept of *social contagion* is a paradigm used to model such phenomena: building on an analogy between the spread of contagious diseases and of ideas, several authors used modified epidemiological models to study social phenomenon as diverse as the spread of information in a network [37], the diffusion of innovations [2, 7, 11] or the outbreaks of riots [8, 9].

The key idea behind the analogy in the *social contagion* paradigm is that the adoption of a new opinion by an individual occurs *via* interactions with others, much like the transmission of a disease through physical contact. While this analogy has several limitations (e.g., the adoption of a new opinion can be a conscious act, may require repeated voluntary interactions, or involve active efforts), it remains compelling for two reasons. First, it offers mathematicians a range of new models that exhibit behaviors not observed in traditional epidemiological or biological models. Second, these models facilitate quantitative analysis, enabling comparisons with real-world data; see, for instance, [7, 8].

Most models in the literature focus on the spread of *a single* opinion. However, the capacity of human populations to accumulate opinions and ideas is widely recognized as a

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key factor in the evolution of ideologies. For sociological insights on this topic, see [32] and references therein.

In this paper, we introduce and analyze a reaction-diffusion model that describes the spread of opinions[1] within a population, where individuals can gradually adopt increasingly *complex* opinions over time. We focus on the following questions: How do different opinions propagate? And is there a limit to the complexity of opinions that can spread?

To formalize our approach, we introduce a general reaction-diffusion model presented below as equation (1.1), which is built on the following hypotheses:

1. The set of all possible opinions is discrete and indexed by $\mathbb{N}$.

2. We consider a closed population: no births or deaths occur within the time scales we consider.

3. The model incorporates a *spatial structure*: the population is distributed across a domain referred to as "space". This space may represent a geographic region where the population resides or a social network through which individuals interact. For simplicity, we assume that the space is the entire real line $\mathbb{R}$.

4. Each individual can exist in one of two states:

   - *Quiet state*: the individual remains static and is receptive to adopting new opinions from others.
   - *Excited state*: the individual is moving and capable of transmitting its own opinion to others.

   We denote by $S_n(t,x)$ and $I_n(t,x)$ the densities of quiet and excited individuals, respectively, holding opinion $n \in \mathbb{N}$ at time $t > 0$, and located at point $x \in \mathbb{R}$.[2]

5. When a quiet individual with opinion n encounters an excited one with opinion k, the quiet individual may change its opinion from n to k. We model this process using a law of mass action, assuming that the rate of this opinion adoption at time t and location x is given by $\alpha(k,n)I_k(t,x)S_n(t,x)$, where $\alpha(k,n) \geq 0$ represents the likelihood of opinion transmission. If $\alpha(k,n) = 0$, it indicates that opinion k cannot be adopted by individuals with opinion n.

6. Once a quiet individual adopts a new opinion, it transitions to the excited state. He can then move and transmit its new opinion to the quiet individuals he would meet. This excited state persists for a certain duration before the individual returns to the quiet state. We denote $\mu_n > 0$ the rate at which an individual holding opinion n ceases to be excited and reverts to the quiet state. Consequently, the average duration that an excited individual with opinion n remains active in promoting its newly adopted idea is given by $1/\mu_n$.

[1] We use the terms *opinions* and *ideas* interchangeably. A more precise term might be *social trait*, referring to a value transmitted between individuals through social interactions. This concept is related to the notion of *meme*, introduced by Dawkins in [12].

[2] The choice of the letters S and I is explained later in Section 1.3, and comes from the analogy with epidemiological (Susceptible-Infected-Recovered) models.

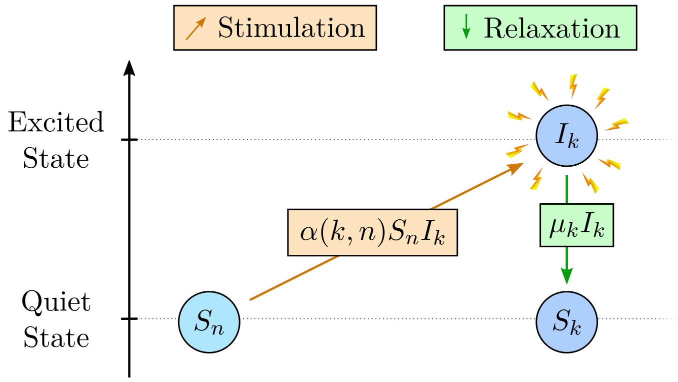

Figure I — *Illustration of the opinion transmission process described in the items 4.-5.-6. above.*

7. When individuals are in the excited state, they move randomly in space, following *independent Brownian motions*. This movement is modeled using Laplace operators at the macroscopic scale, driving the spatial propagation of opinions.[3].

Finally, we define the *opinion graph* $\mathscr{G}$ as the directed graph whose nodes are the elements of $\mathbb{N}$ (representing the opinions) and whose edges are $\{(n,k) \in \mathbb{N}^2 \,:\, \alpha(k,n) > 0\}$. This means that an ordered pair of nodes (n,k) is connected if and only if the opinion k can be adopted by individuals having the opinion n.

By expressing this in the form of equations, we arrive at the following system:

$$\begin{cases} \partial_t S_n = -\left(\sum\limits_{k\in\mathbb{N}} \alpha(k,n) I_k\right) S_n + \mu_n I_n, & n\in\mathbb{N}, \quad t>0, \quad x\in\mathbb{R}, \\ \partial_t I_n = d_n \Delta I_n + \left(\sum\limits_{k\in\mathbb{N}} \alpha(n,k) S_k\right) I_n - \mu_n I_n, & n\in\mathbb{N}, \quad t>0, \quad x\in\mathbb{R}. \end{cases} \tag{1.1}$$

This model is rather general, and without further assumptions on the graph $\mathscr{G}$, it can exhibit many different behaviors. In the present paper, we focus our study on the phenomenon of *accumulation* of opinions rather than on the process of *diversification*: to do so, we shall assume some specific shape on the graph $\mathscr{G}$ that we explain just after. For now, let us mention that the model (1.1) was studied in two situations:

- When the opinion graph $\mathscr{G}$ is made of two nodes, that is, when there are only two opinions, say 0 and 1, and the individuals with opinion 0 can adopt the opinion 1. This situation is equivalent to the classical SIR model of Kermack and McKendrick [27], we detail this in Section 1.3, as it will greatly help to build the intuition for our case.

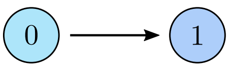

Figure II — *An example of the opinion graph $\mathscr{G}$ for the original SIR model.*

[3] Alternative approaches incorporate nonlocal interactions instead of individual diffusion [13, 14]. We reserve the exploration of such insights for future works.

- When the opinion graph $\mathscr{G}$ is a tree made of one "root node" (say 0), and N "leaf nodes" connected to 0 — see Figure III below. This represents the situation where there is one "neutral" opinion (which cannot be transmitted), while the N other opinions can only be transmitted to individuals holding the neutral opinion. In this configuration, the N different opinions are in competition. The first author showed in [15] that this model exhibits a *selection* phenomenon: only a subset of all the possible opinions spreads.

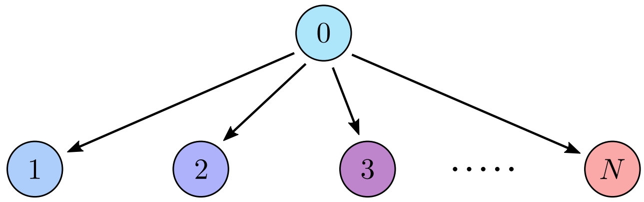

Figure III *— An example of the opinion graph $\mathscr{G}$ for the competition of N opinions.*

In this paper, we study a new instance of (1.1) focused on the accumulation of opinions, by assuming a specific structure on the opinion graph $\mathscr{G}$ — see Figure IV below.

1.2 The model we consider

Considering a general opinion graph $\mathscr{G}$ in (1.1) leads to rather complex behaviors. A first natural restriction is to consider acyclic graph, that is, graphs that contain no loops. Mathematically, this means that for any sequence of integer $(n_1, \cdots, n_p) \in \mathbb{N}^p$, we have $\alpha(n_1, n_2)\alpha(n_2, n_3) \cdots \alpha(n_{p-1}, n_p)\alpha(n_p, n_1) = 0$. This is verified in the two situations mentioned above.

We focus here on the situation where the opinion graph $\mathscr{G}$ is $\mathbb{N}$ where the oriented edges are the couples of adjacent points $(n, n+1)$ — see Figure IV below. In this setting, an individual can only adopt opinion n if it holds the previous opinion $n-1$. Consequently, the integer n serves as a measure of the *opinion's complexity*: the higher n, the harder it becomes for an individual to acquire opinion n, as it must have acquired all prerequisite opinions 1, 2, $\cdots$, $n-1$ beforehand. We see the first opinion $n = 0$ acts as a "basis" opinion, and assume that all individuals possess this opinion initially.

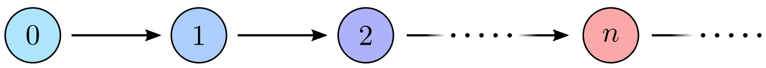

Figure IV *— The opinion graph $\mathscr{G}$ for the accumulation of opinions.*

This setting boils down to assuming, in the general system (1.1), that $\alpha(k, n) = \alpha_k \delta_{k=n+1}$, where $\alpha_k > 0$: only excited individuals with opinion $n+1$ may transmit their opinion with individuals with opinion n.

We are thus led to the following system, which constitutes the main focus of this paper:

$$\begin{cases} \partial_t S_0 = -\alpha_1 S_0 I_1, & t > 0, \quad x \in \mathbb{R}, & (1.2a) \\ \partial_t I_n = d_n \Delta I_n + \alpha_n S_{n-1} I_n - \mu_n I_n, \quad n \in \mathbb{N}^\star, & t > 0, \quad x \in \mathbb{R}, & (1.2b) \\ \partial_t S_n = -\alpha_{n+1} S_n I_{n+1} + \mu_n I_n, \quad n \in \mathbb{N}^\star, & t > 0, \quad x \in \mathbb{R}. & (1.2c) \end{cases}$$

The basis opinion $n = 0$ (the simplest one) cannot be transmitted to other individuals, so there is no need to consider an excited state for $n = 0$. For this reason, we do not consider any function $I_0(t, x)$.

This model is specifically designed to focus on the mechanisms of *accumulation*, or *complexification* of opinions. We believe that, combining the results of the present paper with the results of [15], one would be able to treat the case where the opinion graph $\mathscr{G}$ is a general acyclic oriented graph.

The dynamics of the model are depicted in Figure V below. Initially, all the quiet individuals have opinion $n = 0$ and reside in the S_0 compartment. When a quiet individual with opinion $n = 0$ encounters an excited individual with opinion $n = 1$, the quiet individual can adopt opinion $n = 1$ and transitions from compartment S_0 to I_1. This transition occurs at a rate $\alpha_1 S_0 I_1$. Once in the I_1 compartment, these newly excited individuals remain active for an average duration of $1/\mu_1$, after which they move to the quiet state in compartment S_1, now holding opinion $n = 1$.

The process then continues as such, when a quiet individual with opinion n meets with an excited having opinion $n+1$, he adopts the opinion $n+1$, i.e. it transitions from compartment S_n to I_{n+1} with rate $\alpha_{n+1} S_n I_{n+1}$, resulting in the gradual accumulation of opinions.

Note that interactions only occur between individuals having opinions separated by one degree of complexity (for instance, quiet individuals holding opinion $n = 1$ cannot directly adopt the more complex opinion $n = 3$).

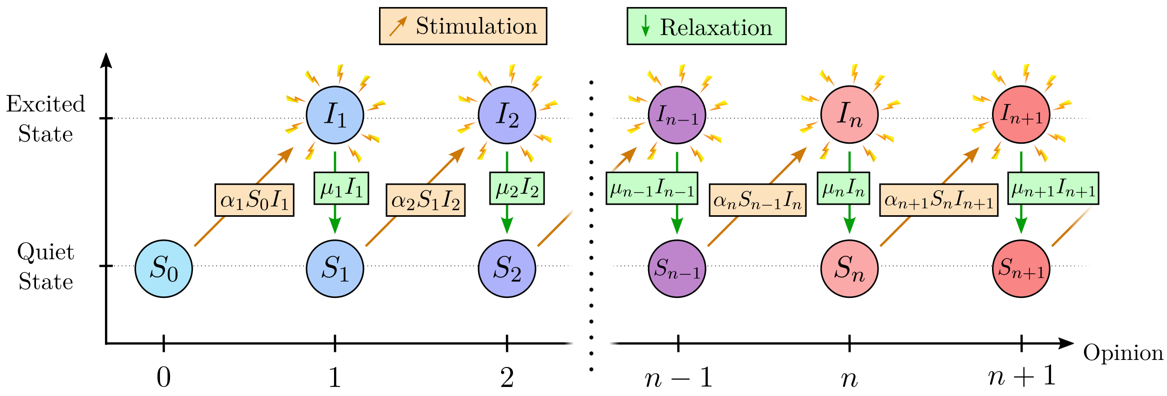

Figure V *— Evolution of the population by transitions through the opinion compartments.*

One of the main questions is the following:

What is the maximal complexity that the population can achieve?

More precisely, starting from a population where all the individuals have the opinion $n = 0$, will we see the emergence of individuals with opinion $n = 1, 2, 3$, etc.? If yes, up to which complexity?

Our main result, Theorem 2.4, completely describes the long-time behavior of the system, and tells us exactly which opinions spread, and with which *speed* (we define this notion below). In particular, depending on the parameters of the system, we give a way to compute the *maximal complexity* that will be reached by the population, that we denote $N^\star \in \mathbb{N} \cup \{+\infty\}$. Although the expression of $N^\star$ is implicit, we give in Theorem 2.5 some qualitative properties of $N^\star$, seen as a function of the system's parameters.

The dynamical system (1.2) must be supplemented with initial conditions, representing the initial distribution of opinions within the population. At initial time $t = 0$, we assume that only the basic opinion $n = 0$ is represented. Specifically, we set

$$S_0(t=0,x) \equiv S_0^\star \in \mathbb{R}_+, \qquad \text{and} \qquad S_n(t=0,x) \equiv 0, \qquad \forall n \in \mathbb{N}^\star.$$

Here, $S_0^\star$ is a nonnegative constant, reflecting a uniform distribution of individuals across space.

To trigger the dynamics, we introduce some *small perturbations* by adding a limited quantity of excited individuals for each opinion. We set

$$I_n(t=0,x) = I_n^0(x), \qquad \forall n \in \mathbb{N}^\star,$$

where the functions $I_n^0 : \mathbb{R} \to \mathbb{R}$ are continuous, non-negative, non-zero and compactly supported. We then investigate which of these perturbations can persist and propagate.

We summarize this initial condition as

$$\begin{cases} S_0|_{t=0} \equiv S_0^\star \in \mathbb{R}_+, & & x \in \mathbb{R}, \\ S_n|_{t=0} \equiv 0, & n \in \mathbb{N}^\star, & x \in \mathbb{R}, \\ I_n|_{t=0} = I_n^0, & n \in \mathbb{N}^\star, & x \in \mathbb{R}. \end{cases} \tag{1.3}$$

1.3 Background: the SIR reaction-diffusion model

As we mentioned earlier, both the general system (1.1) and the particular one (1.2) studied in this paper are inspired by mathematical epidemiology, particularly by the SIR (Susceptible-Infected-Recovered) model introduced by Kermack and McKendrick [27]. In this section, we present the SIR model together with some related results, as it helps to build intuition and fix the notations for our model.

The SIR model describes the spread of a disease in a population, where individuals are divided into three *compartments* (hence the term compartmental models):

- *The susceptibles*: they do not have the disease but are at risk of catching it.
- *The infected*: they are currently infected and can transmit the disease.
- *The recovered*: they have recovered from the disease and are immune, meaning they cannot be reinfected (no waning of immunity).

The density of susceptibles, infected and recovered at time $t > 0$ and position $x \in \mathbb{R}$ are denoted $S(t,x)$, $I(t,x)$, and $R(t,x)$, respectively. When an infected individual encounters a susceptible, the susceptible can become infected according to a law of mass action, that is at a rate αSI, where $\alpha > 0$ represents the likelihood of transmission. Infected individuals then recover at a constant rate $\mu > 0$.

Although the original SIR model did not account for spatial dynamics, this aspect was later incorporated by several authors. The extension of interest here is the one by Källen [26], who assumed that the infected individuals, and only them, diffuse with a diffusivity constant rate $d > 0$. This leads to the following system:

$$\begin{cases} \partial_t S = -\alpha SI, & t > 0, & x \in \mathbb{R}, \\ \partial_t I = d\Delta I + \alpha SI - \mu I, & t > 0, & x \in \mathbb{R}, \\ \partial_t R = \mu I, & t > 0, & x \in \mathbb{R}. \end{cases} \tag{1.4}$$

Remark 1.1 *In [26], the author chose not to include diffusion term on S in* (1.4)*, a decision driven by modeling considerations. When there is such a diffusion term, the analysis is more complex, and some points are still open, we refer to [17] for more details on this topic.*

Here, we follow the same formalism as in the model of Källen: the quiet individual do not diffuse in our model (1.2)*, only the excited ones.*

Observe that up to renaming S as S_0, I as I_1 and R as S_1, the SIR system (1.4) is equivalent to our model (1.2), in the particular case where there are only two opinions. In other words, the SIR model is the simplest instance of the general model (1.2).

The SIR system (1.4) is completed with initial datum $(S^0(x), I^0(x), R^0(x))$, which represents the initial spatial distribution of each group of individuals. We assume $S^0(x) \equiv S^\star \in \mathbb{R}_+$, which means that the susceptibles are initially uniformly distributed across space at the initial time. Next, we assume I^0 is non-negative, non-zero, continuous and compactly supported, reflecting the fact that at the initial time, there are only a few infected individuals, and they are localized in space (in the initial *focus of infection*). Finally, we set $R^0(x) \equiv 0$; while not necessary, this assumption is natural: at the initial time, no one has recovered yet, as the infection is just beginning.

The main result concerning the SIR system (1.4) is the existence of a *threshold effect.* Let $S_\infty(x) := \lim_{t\to+\infty} S(t,x)$, representing the *final density of susceptibles* that remain unaffected after the epidemic, and define the *basic reproduction number* $\mathscr{R}_0 := \frac{\alpha}{\mu} S^\star$.

- If $\mathscr{R}_0 \le 1$, the disease fades away in the sense that

  $$S_\infty(x) \underset{|x|\to+\infty}{\longrightarrow} S^\star,$$

  that is, the final density of susceptible is equal to the initial density, at least for large $|x|$. Hence the disease did not spread. Observe that only the large values of $|x|$ matter: near the initial focus of infection (the support of I^0), some contamination are inevitable, and S_∞ may even be very low in this region, even if the disease does not spread.

- If $\mathscr{R}_0 > 1$, the disease spreads in the sense that there is $S^\dagger \in (0, S_0)$ such that

  $$\sup_{|x|>\delta} |S_\infty(x) - S^\dagger| \underset{\delta\to+\infty}{\longrightarrow} 0,$$

  meaning that the density of susceptible is strictly smaller than the initial density, even far from the initial focus of infection. This indicates that the disease has spread.

  The value $S^\dagger$ is solution of a transcendental equation — see below for further details.

It turns out that we can refine the previous result. When $\mathscr{R}_0 > 1$, we can characterize the *spreading speed* of the epidemic. Define

$$c^\star = 2\sqrt{d(\alpha S^\star - \mu)}\,,$$

then

$$\sup_{ct<|x|} |S - S^\star| \underset{t\to+\infty}{\longrightarrow} 0, \qquad \text{for all } c > c^\star,$$

and

$$\sup_{\delta<|x|<ct} |S - S^\dagger| \underset{\delta,t\to+\infty}{\longrightarrow} 0, \qquad \text{for all } c \in (0, c^\star).$$

The first equation implies that the disease does not spread faster than $c^\star$: an observer moving in one direction with speed $c > c^\star$ will eventually see around him a density of susceptible equal to the initial value $S^\star$, meaning the population there remains unaffected there.

Conversely, the second equation shows that the disease propagates at least as fast as $c^\star$: an observer moving at speed $c < c^\star$ will observe around him a density of susceptible equal to $S^\dagger$ (which is strictly smaller than $S^\star$), meaning the population has already been affected before the observer arrives. Thus, the disease spreads faster than the observer. We say that the function $S(t,x)$ connects $S^\star$ to $S^\dagger$ with spreading speed $c^\star$.

The propagation of the epidemic is illustrated in Figure VI below, which depicts the situation at some time $t > 0$: the population of susceptibles forms two interfaces, while the infected population forms two bumps, both traveling left and right at speed $c^\star$. Meanwhile, the recovered population forms a front traveling at the same speed.

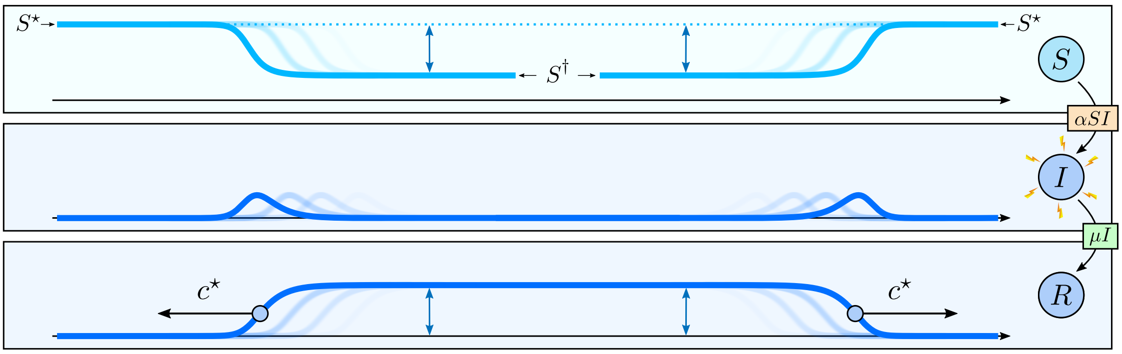

Figure VI — *The SIR reaction-diffusion model in the case of propagation of the disease ($\mathscr{R}_0 > 1$).*

Let us explain how these results are derived, as it will provide insight into the upcoming proofs. The key idea is to observe that the function R satisfies the equation

$$\partial_t R = d\Delta R + \mu S^\star(1 - e^{-\frac{\alpha}{\mu}R}) - \mu R + \mu I_0, \qquad t > 0,\ x \in \mathbb{R}. \tag{1.5}$$

This comes from the following computations. First, observe that we can integrate the equation for S in (1.4) to obtain

$$S(t,x) = S^\star e^{-\frac{\alpha}{\mu}R(t,x)}.$$

Substituting this in the equation for I, we get

$$\partial_t I = d\Delta I + \alpha S^\star e^{-\frac{\alpha}{\mu}R(t,x)} I - \mu I.$$

Now, multiplying by μ and using the fact that $\partial_t R = \mu I$ yields

$$\partial_{tt} R = d\partial_t \Delta R + \alpha S^\star e^{-\frac{\alpha}{\mu}R(t,x)} \partial_t R - \mu \partial_t R.$$

By integrating over t, we arrive at equation (1.5).

Equation (1.5) is a reaction-diffusion equation whose reaction term

$$f(z) = \mu S^\star (1 - e^{-\frac{\alpha}{\mu} z}) - \mu z$$

is concave. Such equations are referred to as *KPP reaction-diffusion equation*, named after Kolmogorov, Petrovski and Piskunov, who studied them in [30]. They proved that, if $f'(0) > 0$, then the solution of the equation propagates toward the unique positive stationary equilibrium of the equation at speed $2\sqrt{df'(0)}$. In our case, $f'(0) = \alpha S^\star - \mu$, which gives the desired result.

For further details on reaction-diffusion equations, we refer to [1, 30]. A standard approach to prove these results involves comparing the solution R of (1.5) with subsolutions and supersolutions, which is the main strategy we employ in this paper.

2 Notations and main results

Before stating our results, in order to justify our notations, we give a heuristic explanation of the dynamics of our system (1.2). In Figure VIII, we represent a typical snapshot of the functions S_0, I_1, S_1, I_2, etc., with each function displayed on a separate graph for clarity. For simplicity, we only depict the right-hand side of the space, as the situation on the left is symmetric.

At initial time, all individuals are quiet and have opinion $n = 0$ — i.e. they are all in the compartment S_0. Then, we consider small perturbations. Specifically, for each $n \in \mathbb{N}^\star$, a small group of localized excited individuals with opinion n is introduced, with their densities denoted by I_n^0.

Let us start to have a look at the the equations governing the populations S_0, I_1, S_1:

$$\begin{cases} \partial_t S_0 = -\alpha_1 S_0 I_1, & t > 0, \quad x \in \mathbb{R}, \\ \partial_t I_1 = d_1 \Delta I_1 + \alpha_1 S_0 I_1 - \mu_1 I_1, & t > 0, \quad x \in \mathbb{R}, \\ \partial_t S_1 = \mu_1 I_1 - \alpha_2 S_1 I_2, & t > 0, \quad x \in \mathbb{R}. \end{cases} \tag{2.1}$$

This system looks like the SIR model, where S_0, I_1, S_1 play the role of the susceptible, infected and recovered respectively, but with a removal term $-\alpha_2 S_1 I_2$. If we disregard this removal term, it is natural to expect, similarly to the SIR system (see subsection 1.3), that when the basic reproduction number $\mathscr{R}_1 := \frac{\alpha_1}{\mu_1} S_0^\star$ exceeds 1, the density $S_0(t,x)$ will decrease from its initial value $S_0^\star$ towards some value $S_0^\dagger > 0$. This decrease is expected to occur by the formation of an interface that moves to the right and to the left at a speed $c_1 := 2\sqrt{d_1(\alpha_1 S_0^\star - \mu_1)}$, as illustrated in the first line of Figure VIII. Meanwhile, the function I_1 forms a traveling bump that propagates at the same speed, shown in the second line of the figure.

As I_1 propagates and becomes quiet, the population S_1 emerges, forming a front that connects 0 to some value $S_1^\star$, as shown in the third line of Figure VIII. However, unlike in the SIR system, the S_1 individuals can still be "contaminated" by the I_2 population afterwards.

At the initial time $t = 0$, there is no population S_1. As a result, the population I_2 has no one to influence, so it simply decays and does not propagate. However, as the population S_1 begins to emerge, the population I_2 will find individuals to convince. More precisely, if we wait long enough for $S_1(t, x)$ to grow towards $S_1^\star$ over a sufficiently large region, then the population I_2 will encounter a local concentration of $S_1^\star$ individuals (with opinion $n = 1$) to transmit its opinion $n = 2$.

We can then expect that if the new basic reproduction number $\mathscr{R}_2 := \frac{\alpha_2}{\mu_2} S1^\star$ is strictly greater than 1, then the population I_2 will propagate — as shown in the fourth line of Figure VIII — with some speed $c_2 := 2\sqrt{d_2(\alpha_2 S_0^\star - \mu_2)}$. As this happens, the number of individuals holding opinion $n = 1$ will decrease: after initially increasing due to the propagation of I_1, the density $S_1(t, x)$ will decline and stabilize at a new value $S_1^\dagger$, with the interface moving at speed c_2. This is depicted in the third line of Figure VIII, where S_1 first increases towards a plateau $S_1^\star$, approximately in the region $(c_2 t, c_1 t)$, and then decreases to a new plateau $S_1^\dagger$ in the zone $(0, c_2 t)$. We see here that it is natural to assume that $c_2 < c_1$, as otherwise the left-moving interface would catch up with the right-moving one, leading to a degenerate situation.

As the population I_2 propagates and becomes quiet, the population S_2 emerges, forming a front that propagates at speed c_2, as shown in the fifth line of Figure VIII. This emergence then gives the opportunity to the I_3 population to propagate, leading to the appearance of S_3, and this process continues in the same manner.

If the basic reproduction number $\mathscr{R}_{n+1} := \frac{\alpha n+1}{\mu_{n+1}} S_n^\star$ is strictly larger than 1, this enables the population I_{n+1} to propagate and spread at speed $c_{n+1} := 2\sqrt{d_{n+1}(\alpha_{n+1} S_n^\star - \mu_{n+1})}$. As a result, $S_n(t, x)$ will decrease as I_{n+1} propagates: for $|x| < c_{n+1} t$, $S_n(t, x)$ is expected to converge to some $S_{n+1}^\dagger$, while in the region $|x| \in (c_{n+1} t, c_n t)$, we should have $S_n(t, x) \approx S_n^\star$. The dynamics of S_n are illustrated in Figure VII.

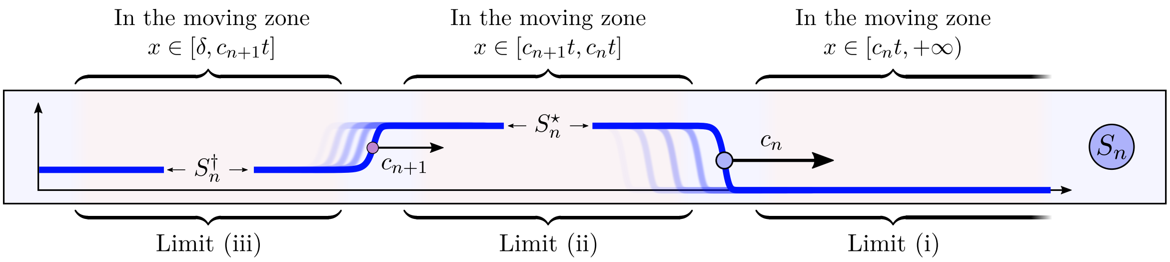

Figure VII — *Asymptotic shape of S_n in the case of propagation ($n \in [\![1, N^\star]\!]$). The limits* (i), (ii) *and* (iii) *refer to those of Theorem 2.4.*

The dynamics may stop at some value of n. Suppose there exists a rank, denoted by $N^\star$, such that the basic reproduction number $\mathscr{R}_{N^\star+1} = \frac{\alpha N^\star+1}{\mu_{N^\star+1}} S_{N^\star}^\star$ is less than or equal to 1. Drawing from the intuition developed from the SIR model, it is natural to think that the population $I_{N^\star+1}$ will not propagate.

As a result, the population $S_{N^\star}$ will converge towards some $S_{N^\star}^\star$ but, since it will not be affected by $I_{N^\star+1}$, it will not experience a decay at the rear. Instead, it will form a simple front, as depicted in the last line of Figure VIII. Consequently, all opinions with indices larger than $N^\star + 1$ will also fail to propagate.

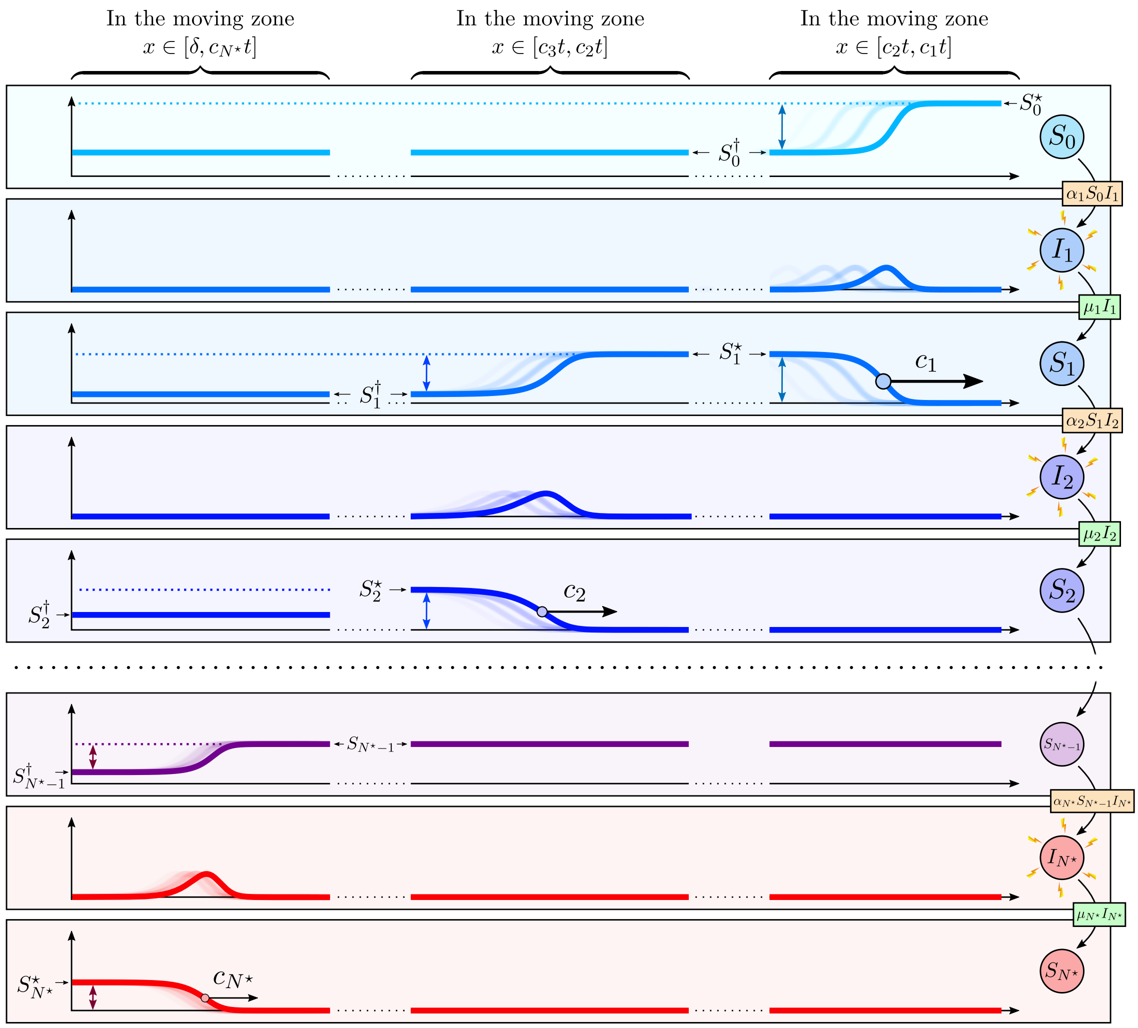

Figure VIII — *Typical propagation dynamics generated by system* (1.2).

The above heuristics lead us to introduce the following notations.

Definition 2.1 (Propagation sequences) *Let $S_0^\star$ and $(d_n, \alpha_n, \mu_n)_{n\in\mathbb{N}^\star}$ be fixed and strictly positive.*

We define $N^\star \in \mathbb{N} \cup \{+\infty\}$ and the propagation sequences $(S_n^\star)_{n\in[\![0,N^\star]\!]}$, $(c_n)_{n\in[\![1,N^\star]\!]}$, and $(S_n^\dagger)_{n\in\mathbb{N}}$ as follows.

For each $n \in \mathbb{N}$ such that $S_n^\star$ is defined ($n = 0$ is given by $S_0^\star$), consider the basic reproduction number

$$\mathcal{R}_{n+1} := \frac{\alpha_{n+1}}{\mu_{n+1}} S_n^\star.$$

- *If $\mathcal{R}_{n+1} \leq 1$, the sequence terminates and we set $N^\star := n$.*
- *If $\mathcal{R}_{n+1} > 1$, we set $S_{n+1}^\star$ to be the the unique positive solution to*

$$S_n^\star \left(1 - e^{-\frac{\alpha_{n+1}}{\mu_{n+1}} S_{n+1}^\star}\right) = S_{n+1}^\star. \tag{2.2}$$

If the sequence $(S_n^\star)_n$ *is not finite, we set* $N^\star = +\infty$. *In addition, we define*

$$c_n := 2\sqrt{d_n(\alpha_n S_{n-1}^\star - \mu_n)}, \qquad \textit{for } n \in [\![1, N^\star]\!],$$

and

$$S_n^\dagger := \begin{cases} S_n^\star - S_{n+1}^\star & \textit{if } n \in [\![0, N^\star - 1]\!], \\ S_n^\star & \textit{if } n = N^\star, \\ 0 & \textit{if } n > N^\star. \end{cases}$$

Using these notations, we can state our main result, which requires, as explained above, the following assumption.

Assumption 2.2 *The sequence* $(c_n)_{n\in[\![1,N^\star]\!]}$ *is strictly decreasing.*

Remark 2.3 *Assumption 2.2 is automatically satisfied if all the parameters* $(d_n, \alpha_n, \mu_n)_{n\in\mathbb{N}}$ *are independent of* n *— i.e.* $(d_n, \alpha_n, \mu_n) = (d, \alpha, \mu)$ *for any* n. *From a modeling perspective, this assumption is quite natural as it implies that more complex opinions propagate slower.*

We are now in position to state our main result. For the sake of readability, in the next theorem, we define $c_0 = +\infty$ and $c_{N^\star+1} = 0$ (when $N^\star$ is finite).

Theorem 2.4 (Long-term behavior of system (1.2))
Let $(S_0, (I_n, S_n)_{n\in\mathbb{N}^\star})$ *be the solution to* (1.2)-(1.3) *and assume that* $S_0^\star$ *and* $(d_n, \alpha_n, \mu_n)_{n\in\mathbb{N}^\star}$ *are such that Assumption 2.2 holds. Then we have*

- *Propagation of the* $N^\star$ *first opinions: for all* $\varepsilon > 0$,

$$\sup_{|x|>(c_n+\varepsilon)t} |S_n(t,x)| \underset{t\to+\infty}{\longrightarrow} 0, \qquad \forall n \in [\![1, N^\star]\!], \tag{i}$$

$$\sup_{(c_{n+1}+\varepsilon)t<|x|<(c_n-\varepsilon)t} |S_n(t,x) - S_n^\star| \underset{t\to+\infty}{\longrightarrow} 0, \qquad \forall n \in [\![0, N^\star]\!], \tag{ii}$$

$$\sup_{\delta<|x|<(c_{n+1}-\varepsilon)t} |S_n(t,x) - S_n^\dagger| \underset{\delta,t\to+\infty}{\longrightarrow} 0, \qquad \forall n \in [\![0, N^\star - 1]\!]. \tag{iii}$$

- *Vanishing of any opinion with complexity strictly higher than* $N^\star$:

$$\sup_{\delta<|x|}\sup_{t>0} |S_n(t,x)| \underset{\delta\to+\infty}{\longrightarrow} 0, \qquad \forall n > N^\star. \tag{iv}$$

This result confirms that the above heuristic is valid. Let us state some remarks.

- $N^\star \in \mathbb{N} \cup \{+\infty\}$ represents the *maximal complexity* of the opinion that can be adopted by the population.

- If $N^\star = 0$, it means that no opinion propagates. In this case, the sequence $(S_n^\star)_n$ consists of only one element, the sequences $(S_n^\dagger)_n$ is empty, and only the lines (ii) (with $c_0 = +\infty$ and $c_1 = 0$) and (iv) need to be considered.

- The propagation of the populations S_n varies depending on whether $n = 0$, $n \in [\![2, N^\star - 1]\!]$, or $n = N^\star$. When $n = 0$, the population S_0 transitions from a non-zero value $S_0^\star$ to another value $S_0^\dagger$. For $n \in [\![2, N^\star - 1]\!]$, S_n connects 0 to some state $S_n^\star$, and then to another state $S_n^\dagger$. Finally, for $n = N^\star$, the population $S_{N^\star}$ connects 0 to some $S_{N^\star}^\star$.

- We set $c_{N^\star+1} = 0$ to ensure that equation (ii) remains valid. However, this can be refined, as we will actually prove:

$$\sup_{\delta<|x|<(c_n-\varepsilon)t} |S_{N^\star}(t,x) - S_{N^\star}^\star| \underset{\delta,t\to+\infty}{\longrightarrow} 0.$$

- A consequence of the theorem is that $\lim_{t\to+\infty} S_n(t,x) \approx S_n^\dagger$, at least for large $|x|$. This means that $S_n^\dagger$ represents the number of individuals who eventually settle for the opinion n.

An important interest of models such as the one considered here is that one can hope to obtain *qualitative properties*. In this paper, we are particularly interested in understanding the relationship between the maximal complexity, the size of the population and the strength of the interactions. To do so, we study how the different parameters of the model influence $N^\star$, the maximal complexity[4].

Theorem 2.5 (Qualitative properties of the maximal complexity $N^\star$) *Let $N^\star \in \mathbb{N} \cup \{+\infty\}$ be as defined in Definition 2.1.*

$\boxed{1}$ **Monotony of $N^\star$.** *Let us take two set of parameters $S_0^\star, (d_n)_{n\in\mathbb{N}^\star}, (\alpha_n)_{n\in\mathbb{N}^\star}, (\mu_n)_{n\in\mathbb{N}^\star}$ and $\overline{S_0^\star}, (\overline{d_n})_{n\in\mathbb{N}^\star}, (\overline{\alpha_n})_{n\in\mathbb{N}^\star}, (\overline{\mu_n})_{n\in\mathbb{N}^\star}$ such that the hypothesis* (2.2) *is verified for each.*

If $S_0^\star \le \overline{S_0^\star}$, $\alpha_n \le \overline{\alpha_n}$ and $\mu_n \ge \overline{\mu_n}$ for all $n \in \mathbb{N}^\star$, then

$$N^\star(S_0^\star, \alpha_i, \mu_i) \le N^\star(\overline{S_0^\star}, \overline{\alpha_i}, \overline{\mu_i}).$$

$\boxed{2}$ **Possibility to reach infinite complexity.** *There are values of the parameters $(d_n)_{n\in\mathbb{N}^\star}, (\alpha_n)_{n\in\mathbb{N}^\star}, (\mu_n)_{n\in\mathbb{N}^\star}$ for which $N^\star = +\infty$. Moreover, for any $\varepsilon > 0$, the parameters can be chosen such that*

$$\lim_{n\to+\infty} S_n^\star \ge S_0^\star - \varepsilon.$$

$\boxed{3}$ **Asymptotic expression of $N^\star$.** *If the parameters are independent of n, meaning there exist constants $d, \alpha, \mu > 0$ such that for all $n \in \mathbb{N}$, we have $d_n = d$, $\alpha_n = \alpha$, and $\mu_n = \mu$, then for large initial populations $S_0^\star$, the following asymptotic equivalent holds:*

$$N^\star(S_0^\star) \underset{S_0^\star\to+\infty}{\sim} \frac{e^{\frac{\alpha}{\mu}S_0^\star}}{\frac{\alpha}{\mu}S_0^\star}. \tag{2.3}$$

[4]For clarity, when needed, we emphasize the dependence of $N^\star$ with respect to the parameters by writing it as a function of $S_0^\star$, $(d_n)_{n\in\mathbb{N}^\star}$, $(\alpha_n)_{n\in\mathbb{N}^\star}$ and $(\mu_n)_{n\in\mathbb{N}^\star}$.

Let us make some remarks on these points. The first one tells us that the quantity $N^\star$ is a non-decreasing function of the initial population $S_0^\star$ and of the transmission parameters $(\alpha_n)_{n\in\mathbb{N}^\star}$ and a non-increasing function of the recovery parameters $(\mu_n)_{n\in\mathbb{N}}$. This is somewhat natural: the larger the α_n, the easier it is for individuals to pass their opinions, the smaller the μ_n, the longer they transmit their opinion. The fact that $N^\star$ is increasing with respect to the size of the population is also natural: larger populations should indeed establish more connections and would be capable of sharing more complex opinions (the dependence on the size of the population is made more explicit in the third point).

The second point indicates that the population can indeed reach opinions with arbitrarily high complexity, and it is even possible for *almost all* the initial population to reach arbitrarily large opinions. However, as we shall see in the proof, this requires the coefficients $(\alpha_n)_{n\in\mathbb{N}^\star}, (\mu_n)_{n\in\mathbb{N}^\star}$ to be chosen carefully.

Although the maximal complexity $N^\star$ is an implicit function of the parameters, the third point tells us that, when the coefficients do not depend on n, $N^\star$ increases almost exponentially with the size of the initial population.

This third point can also be expressed using the basic reproduction number $\mathscr{R}_0 = \frac{\alpha S_0^\star}{\mu}$:

$$N^\star(\mathscr{R}_0) \underset{\mathscr{R}_0\to+\infty}{\sim} \frac{e^{\mathscr{R}_0}}{\mathscr{R}_0}.$$

From the modeling point of view, the rapid growth in complexity predicted by the third point seems rather natural. One can indeed expect that large populations should create more interpersonal connections, leading to more interactions, and this should result in higher complexity and diversity of opinions. We refer to [32] for related discussions.

To illustrate how the initial density $S_0^\star$ influences the dynamics of the system, let us plot the graph of the function

$$(S_0^\star, n) \in \mathbb{R}_+ \times \mathbb{N} \mapsto \frac{S_n^\dagger}{S_0^\star}.$$

This quantity represents the asymptotic proportion of individuals holding opinion n, relative to the initial population size $S_0^\star$, or, to state it differently, is represents the proportion of individuals who eventually adopt the n-th opinion[5]. Because the total number of individuals is conserved, we have $\sum_n \frac{S_n^\dagger}{S_0^\star} = 1$.

If $n \leq N^\star$, the quantity $\frac{S_n^\dagger}{S_0^\star}$ is strictly positive, and for $n > N^\star$, this quantity is zero (no individuals adopt these opinions).

In the following graph, the intensity of shaded areas indicates the prevalence of opinion n within the population in long time — darker shades represent higher proportions.

[5]See the last point below Theorem 2.4.

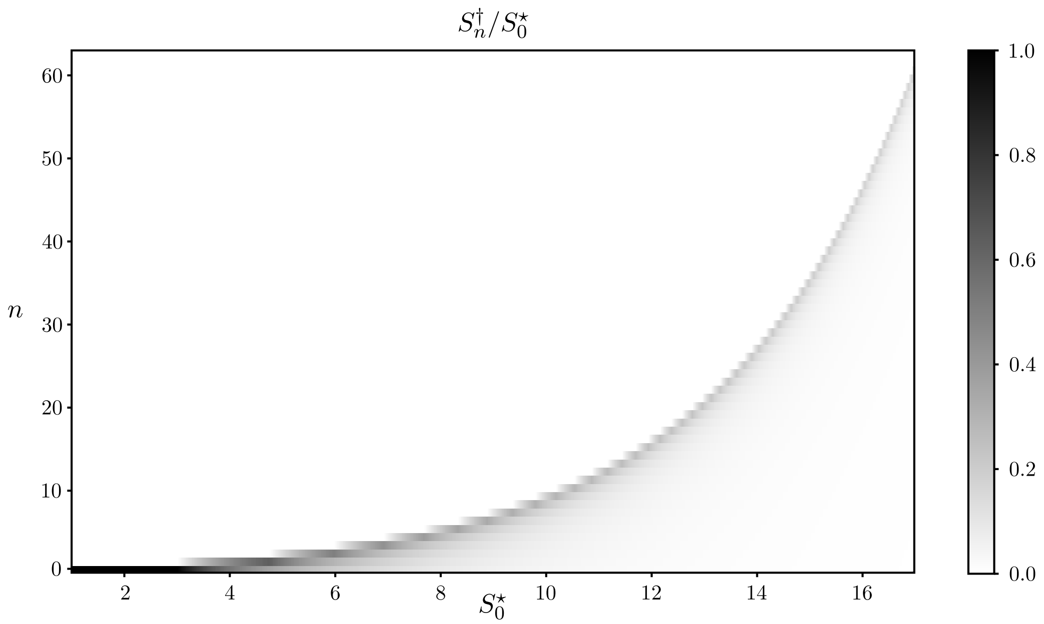

Figure IX — *Asymptotic proportion $S_n^\dagger/S_0^\star$ of individuals holding opinion n.*

We observe that the maximal opinion complexity $N^\star = N^\star(S_0^\star)$ corresponds to the highest point of non-zero proportion along each column (for $n \leq N^\star$, the area is gray, while for $n > N^\star$ the zone is blank). Tracking the curve formed by these maximal points reveals an almost exponential shape, consistent with the predicted behavior of $N^\star$ for large $S_0^\star$, as described in the final point of Theorem 2.5 — see (2.3).

3 Proof of Theorem 2.4

The goal of this section is to prove Theorem 2.4. As explained in Section 1.3, the classical SIR system can be rewritten as a single scalar reaction-diffusion equation for the recovered. Building on this intuition, we introduce the functions R_n, $n \in \mathbb{N}^\star$, defined by

$$\begin{cases} \partial_t R_n = \mu_n I_n, & t > 0, \quad x \in \mathbb{R}, & (3.1a) \\ R_n|_{t=0} \equiv 0, & x \in \mathbb{R}, & (3.1b) \end{cases}$$

and we will study their spreading properties first instead of studying the functions S_n and I_n. Unlike for the classical SIR system, the functions R_n will not satisfy a scalar reaction-diffusion equations, nor a “simple” reaction-diffusion system: they will be be coupled, in a somewhat implicit fashion. The core of the proof will be to control the influence of each R_n on the others.

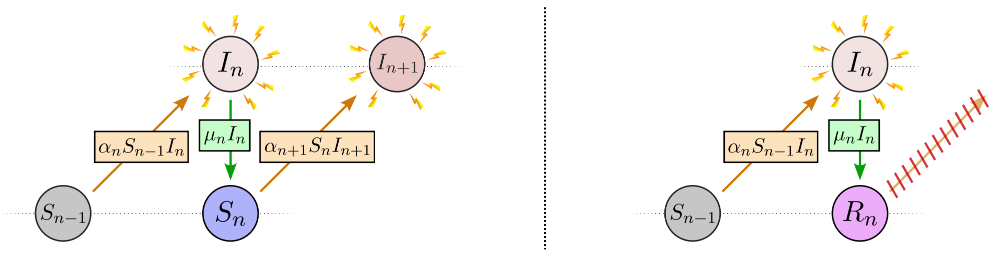

Figure X — S_n vs. R_n.

Our key result concerning the functions R_n is the following.

Proposition 3.1 (Propagation of R_n) *Assume that the hypotheses of Theorem 2.4 hold true. Then, for every $n \in [\![1, N^\star]\!]$, we have*

$$\sup_{ct<|x|} |R_n(t,x)| \underset{t\to+\infty}{\longrightarrow} 0, \qquad \forall c > c_n, \tag{j}$$

and

$$\sup_{\delta<|x|<ct} |R_n(t,x) - S_n^\star| \underset{\delta,t\to+\infty}{\longrightarrow} 0, \qquad \forall c \in (0, c_n). \tag{jj}$$

This proposition tells us that the function R_n spreads toward $S_n^\star$ with speed c_n. The propagation of the function R_n is similar to the propagation we want to prove on the functions S_n, except it does not have the decay toward $S_n^\dagger$ at the back of the front — see Figure XI. For this reason, it will be more convenient to work with the functions R_n rather than S_n.

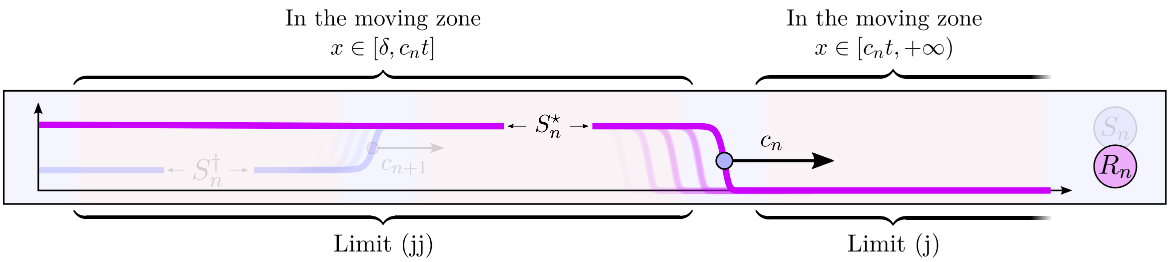

Figure XI — *R_n behave like S_n but without the decay toward $S_n^\dagger$ at the back of the front.*

This section is organized as follows. In Section 3.1, we give some basic estimates on the functions R_n. In Section 3.2 we show that each R_n satisfies a reaction-diffusion equation up to some perturbation term. Then, in Section 3.3 we prove Proposition 3.1. Finally, in Section 3.4, we show how Proposition 3.1 implies Theorem 2.4.

3.1 Basic results on the auxiliary functions R_n

We start this section with a remark concerning the function R_1.

Remark 3.2 *The computations presented in the introduction, at the end of Section 1.3, yield that R_1 satisfies the same reaction-diffusion equation than in the case of the classical SIR system (the functions S_0, I_1, R_1 actually form a SIR system), that is, we have*

$$\partial_t R_1 = d_1 \Delta R_1 + f_1(R_1) + \mu_1 I_1^0, \qquad t > 0,\ x \in \mathbb{R}.$$

Therefore, Proposition 3.1 holds true for $n = 1$. Our proof of Proposition 3.1 will be done by induction and we shall use this as our base case.

The next lemma explains the relationship between the functions R_n and S_n.

Lemma 3.3 (Controlling S_n with R_n) *For all $t > 0$ and $x \in \mathbb{R}$, we have*

$$S_0^\star\, e^{-\frac{\alpha_1}{\mu_1} R_1(t,x)} = S_0(t,x) \le S_0^\star, \tag{3.2}$$

$$R_n(t,x)\, e^{-\frac{\alpha_{n+1}}{\mu_{n+1}} R_{n+1}(t,x)} \le S_n(t,x) \le R_n(t,x), \qquad \forall n \in \mathbb{N}^\star. \tag{3.3}$$

Proof of Lemma 3.3. We start by establishing (3.2). Because S_0 and I_1 are non-negative, it follows from (1.2a) that

$$-\alpha_1 S_0 I_1 = \partial_t S_0 \leq 0, \qquad t > 0,\ x \in \mathbb{R}.$$

Dividing this by S_0 and using the equation for $\partial_t R_1$ from (3.1a), we obtain

$$-\frac{\alpha_1}{\mu_1}\partial_t R_1 = \frac{\partial_t S_0}{S_0} \leq 0.$$

Now we integrate from 0 to t. Recalling that $R_1|_{t=0} \equiv 0$, we get

$$-\frac{\alpha_1}{\mu_1} R_1(t,x) = \ln(S_0(t,x)) - \ln(S_0^\star) \leq 0,$$

which directly leads to (3.2).

For $n \in \mathbb{N}^\star$, to get the upper bound of S_n, we consider equation (1.2c) where S_n and I_{n+1} are both positive. Using (3.1a), this leads to

$$\partial_t S_n \leq \mu_n I_n = \partial_t R_n,$$

Integrating from 0 to t and using that $S_n|_{t=0} = R_n|_{t=0} \equiv 0$, we get

$$S_n(t,x) \leq R_n(t,x) \tag{3.4}$$

which is the upper bound of S_n as specified (3.3).

To get the lower bound in (3.3), we divide (1.2c) by S_n and use (3.1a) to get

$$\frac{\partial_t S_n}{S_n} = -\frac{\alpha_{n+1}}{\mu_{n+1}}\partial_t R_{n+1} + \frac{\partial_t R_n}{S_n}, \qquad t > 0,\ x \in \mathbb{R}.$$

Using the upper bound (3.4) on S_n and the positivity of $\partial_t R_n = \mu_n I_n$, it follows that

$$\frac{\partial_t S_n}{S_n} \geq -\frac{\alpha_{n+1}}{\mu_{n+1}}\partial_t R_{n+1} + \frac{\partial_t R_n}{R_n}. \tag{3.5}$$

We integrate the inequality (3.5) from some small $\eta > 0$ to t. This results in

$$S_n(t,x) \geq \frac{S_n(\eta,x)}{R_n(\eta,x)} \times R_n(t,x) e^{-\frac{\alpha_{n+1}}{\mu_{n+1}} R_{n+1}(t,x)}.$$

Let us now prove that $S_n(\eta,x)/R_n(\eta,x)$ converges to 1 for all x as η goes to zero. By combining (1.2c) and (3.1a), we obtain

$$\partial_t S_n = \partial_t R_n - \alpha_{n+1} S_n I_{n+1}.$$

Integrating from 0 to η and using again that $S_n|_{t=0} = R_n|_{t=0} \equiv 0$, we are led to

$$S_n(\eta,x) = R_n(\eta,x) - \alpha_{n+1}\int_0^\eta S_n(s,x) I_{n+1}(s,x)ds,$$

which can be rearranged as

$$\frac{S_n(\eta,x)}{R_n(\eta,x)} - 1 = \frac{\alpha_{n+1}}{R_n(\eta,x)}\int_0^\eta S_n(s,x) I_{n+1}(s,x)ds. \tag{3.6}$$

Given the positivity and the upper bound on S_n established in (3.4), as well as the positivity of $\partial_t R_n = \mu_n I_n$, we have, for any $s \in (0, \eta)$,

$$0 \leq S_n(s,x) \leq R_n(s,x) \leq R_n(\eta,x).$$

As a result, (3.6) implies that

$$\left|\frac{S_n(\eta,x)}{R_n(\eta,x)} - 1\right| \leq \alpha_{n+1} \times \eta \times \sup_{s\in(0,\eta)} |I_{n+1}(s,x)|$$

which vanishes as η approaches 0. □

The next lemma shows that R_n satisfies some differential inequality involving R_{n-1}. In the Section 3.2, we will improve this result and prove that R_n "almost" satisfies a scalar reaction-diffusion equation.

Lemma 3.4 (R_n is sub-solution to a perturbed Fisher-KPP equation) *For all $n \in \mathbb{N}^\star$, for all $t > 0$ and $x \in \mathbb{R}$, there holds*

$$\partial_t R_n \leq d_n \Delta R_n + \mu_n R_{n-1}\Big(1 - e^{-\frac{\alpha_n}{\mu_n} R_n}\Big) - \mu_n R_n + \mu_n I_n^0. \tag{3.7}$$

Proof of Lemma 3.4. The proof relies on a computation similar to the one presented in Section 1.3 to obtain (1.5): we multiply (1.2b) by μ_n and using the definition of R_n, (3.1a), we find

$$\partial_{tt} R_n = d_n \Delta \partial_t R_n + \mu_n \alpha_n S_{n-1} I_n - \mu_n \partial_t R_n.$$

Integrating from 0 to t and recalling that $R_n|_{t=0} \equiv 0$, we obtain

$$\partial_t R_n - \mu_n I_n^0 = d_n \Delta R_n + \mu_n \int_0^t \alpha_n S_{n-1} I_n \, ds - \mu_n R_n. \tag{3.8}$$

Combining (1.2c) with the definition of R_n, (3.1a), the term under the integral rewrites

$$\alpha_n S_{n-1} I_n = \partial_t (R_{n-1} - S_{n-1}),$$

so that we have

$$\partial_t R_n - \mu_n I_n^0 = d_n \Delta R_n + \mu_n (R_{n-1} - S_{n-1}) - \mu_n R_n. \tag{3.9}$$

Now, using the lower bound (3.3) for S_n given in Lemma 3.3, we find

$$\partial_t R_n \leq d_n \Delta R_n + \mu_n \left(R_{n-1} - R_{n-1}\, e^{-\frac{\alpha_n}{\mu_n} R_n}\right) - \mu_n R_n + \mu_n I_n^0,$$

which provides (3.7). □

A consequence of Lemma 3.4 is that the functions S_n, I_n, R_n are uniformly bounded.

Lemma 3.5 (Uniform upper bounds on S_n, I_n and R_n) *For all $n \in \mathbb{N}$, there is $K_n > 0$ such that, for all $t > 0$ and $x \in \mathbb{R}$,*

$$S_n(t,x) + I_n(t,x) + R_n(t,x) \leq K_n.$$

Proof of Lemma 3.5.

• *Upper bound on R_n.* We argue by induction. If $n = 1$, then as explained in Remark 3.2, we have $\partial_t R_1 = d_1 \Delta R_1 + f_1(R_1) + \mu_1 I_1^0$, and $R_1(0, \cdot) \equiv 0$. We can take $C_1 > 0$ such that $0 \geq f_1(C_1) + \mu_1 I_1^0$, so that C_1 is supersolution to the equation that R_1 solves. Hence, by the parabolic comparison principle, $R_1 \leq C_1$.

Now, for $n \in \mathbb{N}^\star$, $n > 1$, assume that there is $C_{n-1} > 0$ such that $R_{n-1} \leq C_{n-1}$. Hence, because R_n satisfies (3.7) from Lemma 3.4, we have

$$\partial_t R_n \leq d_n \Delta R_n + \mu_n C_{n-1}(1 - e^{-\frac{\alpha_n}{\mu_n} R_n}) - \mu_n R_n + \mu_n I_n^0.$$

Because I_n^0 is bounded, using the parabolic comparison principle, we can find then a constant $C_n > 0$ large enough to be a supersolution of the above equation. This gives $R_n \leq C_n$ for all $t > 0$ and $x \in \mathbb{R}$. By induction, each R_n is therefore uniformly bounded.

• *Upper bound on S_n.* Combining Lemma 3.3 and the previous point, we have $S_n \leq R_n \leq C_n$ for all $t > 0$ and $x \in \mathbb{R}$.

• *Upper bound on I_n.* For $n \in \mathbb{N}^\star$, we know that I_n satisfies (1.2b), which is a linear parabolic equation with bounded coefficients. Therefore, owing to the parabolic Harnack inequality — see [19, Section 7, Theorem 10] for instance — there is a constant $k_n > 0$ such that for any $t > 1$ and any $x \in \mathbb{R}$, we have

$$I_n(t,x) \leq k_n \inf_{\tau \in [t+1, t+2]} I_n(\tau, x).$$

Therefore

$$I_n(t,x) \leq k_n \int_{t+1}^{t+2} I_n(\tau, x)\, d\tau \leq \frac{k_n}{\mu_n} R_n(t+2, x) \leq \frac{k_n}{\mu_n} C_n,$$

hence I_n is also uniformly bounded. □

As explained above, we prove Proposition 3.1 by induction. The induction hypothesis will be denoted $\mathcal{H}_n$ for $n \in [\![1, N^\star]\!]$ and is the following.

$$\mathcal{H}_n : \begin{cases} \sup\limits_{\delta < |x| < ct} |R_n(t,x) - S_n^\star| \xrightarrow[\delta, t \to \infty]{} 0, & \forall c \in (0, c_n), \quad (3.10a) \\[2ex] \sup\limits_{|x| > ct} |R_n(t,x)| \xrightarrow[\delta, t \to \infty]{} 0, & \forall c \in (c_n, +\infty). \quad (3.10b) \end{cases}$$

As observed in Remark 3.2, we already know that $\mathcal{H}_n$ holds true for $n = 1$.

For the sake of clarity, it will useful in some places to adopt the convention that $\mathcal{H}_0$ is a vacuously true hypothesis, meaning it is always satisfied. In other words, when we state "assume that $\mathcal{H}_n$ holds for $n = 0$", nothing is actually being assumed.

Remark 3.6 *It is clear that $R_n(t,x)$ is non-decreasing with respect to t. For any $n \in \mathbb{N}^\star$, if $\mathcal{H}_n$ holds true, the boundedness of R_n given by Lemma 3.5 tells us that there is $R_n^\infty(x)$ such that*

$$R_n(t,x) \underset{t \to \infty}{\nearrow} R_n^\infty(x),$$

and this convergence is a priori *only pointwise — it turns out that it is actually locally uniform, we shall discuss this later.*

In addition, observe that taking the limit $t \to +\infty$ *in* (3.10a), $\mathcal{H}_n$ *also implies that* R_n^∞ *converges to* $S_n^\star$ *when* x *is large, that is*

$$R_n^\infty(x) \underset{|x|\to+\infty}{\longrightarrow} S_n^\star. \tag{3.11}$$

In particular — we shall use this several time in the sequel — this means that there is $\varepsilon_n(x)$ *such that* $\varepsilon_n(x) \underset{|x|\to+\infty}{\longrightarrow} 0$ *and*

$$R_n(t,x) \le S_n^\star + \varepsilon_n(x), \qquad \forall t>0,\, \forall x \in \mathbb{R}.$$

3.2 R_n solves a perturbed Fisher-KPP equation

We define the reaction function f_n for $n \in [\![1, N^\star]\!]$ as

$$f_n(z) := \mu_n \left(S_{n-1}^\star \left(1 - e^{-\frac{\alpha_n}{\mu_n} z}\right) - z \right), \qquad \text{for any } z \in \mathbb{R}. \tag{3.12}$$

The function f_n is a KPP reaction, and $f_n'(0) = \alpha_n S_{n-1}^\star - \mu_n > 0$ for $n \le N^\star$ (owing to our definition of $N^\star$). Moreover observe that, $S_n^\star$ as defined in (2.2) is the unique positive zero of f_n.

The main point of this subsection is to show that R_n satisfies a "perturbed" KPP equation.

Proposition 3.7 (R_n solves a perturbed KPP equation) *Let* $n \in [\![1, N^\star]\!]$ *and assume that* $\mathcal{H}_{n-1}$ *holds true. Then, there is* $\varepsilon_n \in L^\infty(\mathbb{R})$ *such that* $\varepsilon_n(x) \underset{|x|\to+\infty}{\longrightarrow} 0$ *and*

$$-\varepsilon_n \le \partial_t R_n - d_n \Delta R_n - f_n(R_n) \le \varepsilon_n, \tag{3.13}$$

for any $t > 0$ *and any* $x \in \mathbb{R}$.

Remark 3.8 *Proposition 3.7 is the cornerstone for proving the spreading result stated in Proposition 3.1. Indeed, if we had* $\varepsilon_n = 0$ *in* (3.13) *— i.e. if* R_n *would solve exactly* $\partial_t R_n = d_n \Delta R_n + f_n(R_n)$ *—, classical results from reaction-diffusion theory would directly yield that* R_n *spreads with speed* c_n *toward* $S_n^\star$*, that is,* $\mathcal{H}_n$ *would be verified.*

The presence of the perturbation ε_n *rises some technicalities, that will be tackled in the next Section 3.3.*

The proof of Proposition 3.7 relies on Lemma 3.4 above and on the two other lemmas, namely Lemma 3.9 and Lemma 3.11, that we now state and prove.

Lemma 3.9 (Exponential estimates) *Let* $n \in [\![1, N^\star]\!]$ *and assume that the induction hypothesis* $\mathcal{H}_{n-1}$ *holds true. Then we have the following upper bounds on* R_n *and* I_n.

For all $c > c_n$*, there are* $\Lambda_n, \tilde{\Lambda}_n, \lambda_n > 0$ *such that*

$$I_n(t,x) \le \Lambda_n\, e^{-\lambda_n(x-ct)}, \qquad \forall t>0,\, \forall x \in \mathbb{R}, \tag{3.14}$$

and

$$R_n(t,x) \le \tilde{\Lambda}_n\, e^{-\lambda_n(x-ct)}, \qquad \forall t>0,\, \forall x \in \mathbb{R}. \tag{3.15}$$

Proof of Lemma 3.9.

• *Proof of* (3.14). Let $c > c_n$ be fixed. By definition of c_n, we can find $\varepsilon > 0$ small enough so that

$$c > 2\sqrt{d_n(\alpha_n(S^\star_{n-1}+\varepsilon)-\mu_n)}\,.$$

From (3.3), we have

$$\partial_t I_n = d_n\Delta I_n + (\alpha_n S_{n-1}-\mu_n)I_n \leq d_n\Delta I_n + (\alpha_n R_{n-1}-\mu_n)I_n, \qquad \forall t>0,\, \forall x\in\mathbb{R}.$$

Moreover, according to Remark 3.6, since we assume that $\mathcal{H}_{n-1}$ holds, we can take $R > 0$ large enough so that, for any $x > R$, we have $R_{n-1}(t,x) \leq S^\star_{n-1}+\varepsilon$ (where the ε has been chosen above). As a result,

$$\partial_t I_n \leq d_n\Delta I_n + (\alpha_n(S^\star_{n-1}+\varepsilon)-\mu_n)I_n, \qquad \forall t>0,\, \forall x>R.$$

Define the function $\overline{I_n}(t,x) := \Lambda_n e^{-\lambda_n(x-ct)}$, where Λ_n and λ_n are two positive constants, chosen such that

$$c\lambda_n - d_n\lambda_n^2 - \big(\alpha_n(S^\star_{n-1}+\varepsilon)-\mu_n\big) \geq 0,$$

(this is possible because the discriminant of this equation is positive), and with Λ_n chosen sufficiently large to ensure that, at initial time,

$$\overline{I_n}|_{t=0}(x) \;=\; \Lambda_n e^{-\lambda_n x} \;\geq\; I_n|_{t=0}(x), \qquad \text{for all } x>R,$$

and that, for all $t > 0$,

$$\overline{I_n}(t,R) \;=\; \Lambda_n e^{-\lambda_n(R-ct)} \;\geq\; \Lambda_n e^{-\lambda_n R} \;\geq\; \sup_{t>0}\sup_{x\in\mathbb{R}} I_n(t,x). \tag{3.16}$$

Therefore, the function $\overline{I_n}$ satisfies $\partial_t\overline{I_n} \geq d_n\Delta\overline{I_n} + (\alpha_n(S^\star_{n-1}+\varepsilon)-\mu_n)\overline{I_n}$ for $t > 0$ and $x > R$, and $\overline{I_n}(0,\cdot) \geq I_n(0,\cdot)$ and $\overline{I_n}(t,R) \geq I_n(t,R)$. Hence, the parabolic comparison principle (applied on the set $(0,+\infty)\times(R,+\infty)$ to $\overline{I_n}, I_n$) yields $\overline{I_n} \geq I_n$ for any $t > 0$ and $x > R$.

Now, for $x \leq R$ and $t > 0$, we have, using Lemma 3.5,

$$\overline{I_n}(t,x) \;\geq\; \Lambda_n e^{-\lambda_n R} \overset{(3.16)}{\geq} \sup_{t>0}\sup_{x\in\mathbb{R}} I_n(t,x),$$

and this completes the proof of (3.14).

• *Proof of* (3.15). In view of (3.1), we multiply (3.14) by μ_n before integrating between 0 to t, this yields

$$R_n(t,x) \leq \frac{\mu_n}{c\lambda_n}\Lambda_n e^{-\lambda_n(x-ct)},$$

for all $t > 0$ and $x \in \mathbb{R}$. This gives (3.15), with $\tilde{\Lambda}_n = \frac{\mu_n}{c\lambda_n}\Lambda_n$. □

Remark 3.10 (Symmetric version of (3.14) and (3.15)) *Using similar arguments, we also find that, for all $t > 0$ and $x \in \mathbb{R}$,* (3.14) *and* (3.15)*,*

$$I_n \leq \Lambda_n\, e^{\lambda_n(x+ct)}, \tag{3.17}$$

and

$$R_n \leq \tilde{\Lambda}_n\, e^{\lambda_n(x+ct)}. \tag{3.18}$$

Lemma 3.11 (R_n is super-solution to a perturbed Fisher-KPP equation) *Assume $N^\star \geq 2$. Let $n \in [\![2, N^\star]\!]$ and assume that the induction hypothesis $\mathcal{H}_{n-1}$ holds true. Then there is $\varepsilon_n \in L^\infty(\mathbb{R})$ such that $\varepsilon_n(x) \xrightarrow[|x|\to+\infty]{} 0$ and*

$$\partial_t R_n \geq d_n \Delta R_n + f_n(R_n) - \varepsilon_n, \qquad \forall t > 0,\, \forall x \in \mathbb{R}. \tag{3.19}$$

Proof of Lemma 3.11. Consider (3.8) as established in the proof of Lemma 3.4. By using the definition of R_n (3.1a), this equality can be recast

$$\partial_t R_n = d_n \Delta R_n + \mu_n \int_0^t \frac{\alpha_n}{\mu_n} S_{n-1}\, \partial_t R_n\, ds - \mu_n R_n + \mu_n I_n^0. \tag{3.20}$$

Let $\tilde{c} = \frac{c_n + c_{n-1}}{2}$ be fixed. Due to the positivity of the integrated term in (3.20) we can write (where the minimum of two reals a and b is denoted $a \wedge b := \min\{a, b\}$)

$$\int_0^t \frac{\alpha_n}{\mu_n} S_{n-1}\, \partial_t R_n\, ds \geq \int_{\frac{|x|}{\tilde{c}} \wedge t}^t \frac{\alpha_n}{\mu_n} S_{n-1}\, \partial_t R_n\, ds, \qquad \forall t > 0,\; x \in \mathbb{R}.$$

Using (3.3) from Lemma 3.3, we get

$$\int_0^t \frac{\alpha_n}{\mu_n} S_{n-1}\, \partial_t R_n\, ds \geq \int_{\frac{|x|}{\tilde{c}} \wedge t}^t \frac{\alpha_n}{\mu_n} R_{n-1}\, e^{-\frac{\alpha_n}{\mu_n} R_n}\, \partial_t R_n\, ds,$$

and using the fact that $R_{n-1}(t, x)$ is non-decreasing with respect to the t variable yields

$$\begin{aligned}
&\int_0^t \frac{\alpha_n}{\mu_n} S_{n-1}\, \partial_t R_n\, ds \\
&\quad \geq R_{n-1}\Big(\tfrac{|x|}{\tilde{c}} \wedge t, x\Big) \int_{\frac{|x|}{\tilde{c}} \wedge t}^t \frac{\alpha_n}{\mu_n} e^{-\frac{\alpha_n}{\mu_n} R_n}\, \partial_t R_n\, ds \\
&\quad = R_{n-1}\Big(\tfrac{|x|}{\tilde{c}} \wedge t, x\Big) \int_{\frac{|x|}{\tilde{c}} \wedge t}^t -\partial_t \left(e^{-\frac{\alpha_n}{\mu_n} R_n}\right) ds \\
&\quad = R_{n-1}\Big(\tfrac{|x|}{\tilde{c}} \wedge t, x\Big) \left[\exp\left(-\frac{\alpha_n}{\mu_n} R_n\Big(\tfrac{|x|}{\tilde{c}} \wedge t, x\Big)\right) - \exp\left(-\frac{\alpha_n}{\mu_n} R_n(t, x)\right)\right] \\
&\quad = R_{n-1}\Big(\tfrac{|x|}{\tilde{c}} \wedge t, x\Big) \left[\left(1 - \exp\left(-\frac{\alpha_n}{\mu_n} R_n(t, x)\right)\right) - \left(1 - \exp\left(-\frac{\alpha_n}{\mu_n} R_n\Big(\tfrac{|x|}{\tilde{c}} \wedge t, x\Big)\right)\right)\right] \\
&\qquad\qquad + S_{n-1}^\star \left(1 - \exp\left(-\frac{\alpha_n}{\mu_n} R_n(t, x)\right)\right) - S_{n-1}^\star \left(1 - \exp\left(-\frac{\alpha_n}{\mu_n} R_n(t, x)\right)\right) \\
&\quad = S_{n-1}^\star \left(1 - \exp\left(-\frac{\alpha_n}{\mu_n} R_n(t, x)\right)\right) - \rho_n^1(t, x) - \rho_n^2(t, x),
\end{aligned} \tag{3.21}$$

where

$$\rho_n^1(t, x) = R_{n-1}\Big(\tfrac{|x|}{\tilde{c}} \wedge t, x\Big) \left[1 - \exp\Big(-\frac{\alpha_n}{\mu_n} R_n\Big(\tfrac{|x|}{\tilde{c}} \wedge t, x\Big)\Big)\right] \tag{3.22}$$

and

$$\rho_n^2(t,x) = \left[S^\star_{n-1} - R_{n-1}\left(\tfrac{|x|}{\tilde{c}} \wedge t, x\right)\right]\left[1 - \exp\left(-\frac{\alpha_n}{\mu_n} R_n(t,x)\right)\right]. \tag{3.23}$$

Combining (3.20) and (3.21), we are eventually led to

$$\partial_t R_n \geq d_n \Delta R_n + f_n(R_n) + \mu_n I_n^0 - \mu_n \rho_n^1(t,x) - \mu_n \rho_n^2(t,x). \tag{3.24}$$

The goal now is to draw estimates from above of $\rho_n^1(t,x)$ and $\rho_n^2(t,x)$.

• *Estimate on* ρ_n^1. Owing to Lemma 3.5, $R_{n-1} \leq K_{n-1}$. Using the concavity of $z \mapsto 1-e^{-z}$ and the fact that R_n is non-decreasing with respect to the t variable, we have

$$\begin{aligned}
\rho_n^1(t,x) &\leq K_{n-1}\left[1 - \exp\left(-\frac{\alpha_n}{\mu_n} R_n\left(\tfrac{|x|}{\tilde{c}} \wedge t, x\right)\right)\right] \\
&\leq K_{n-1}\frac{\alpha_n}{\mu_n} R_n\left(\tfrac{|x|}{\tilde{c}} \wedge t, x\right) \\
&\leq K_{n-1}\frac{\alpha_n}{\mu_n} R_n\left(\frac{|x|}{\tilde{c}}, x\right).
\end{aligned}$$

Now, fixing $c > 0$ such that $c_n < c < \tilde{c} = (c_n + c_{n-1})/2$, (3.15) from Lemma 3.9 gives (for $x > 0$)

$$\rho_n^1(t,x) \leq K_{n-1}\frac{\alpha_n}{\mu_n}\tilde{\Lambda}_n \times \exp\left(-\lambda_n \underbrace{\left(1 - \frac{c}{\tilde{c}}\right)}_{>0} x\right).$$

Similarly, we get, for $x < 0$,

$$\rho_n^1(t,x) \leq K_{n-1}\frac{\alpha_n}{\mu_n}\tilde{\Lambda}_n \times \exp\left(\lambda_n \left(1 - \frac{c}{\tilde{c}}\right) x\right),$$

so that in the end, we have that there are $K, q > 0$ such that

$$\rho_n^1(t,x) \leq K e^{-q|x|}. \tag{3.25}$$

• *Estimate on* ρ_n^2. We write

$$\rho_n^2(t,x) = \rho_n^2(t,x)\mathbb{1}_{|x|<\tilde{c}t} + \rho_n^2(t,x)\mathbb{1}_{|x|\geq\tilde{c}t}.$$

Because S_n, R_n are bounded (thanks to Lemma 3.9), and using the concavity of $z \mapsto 1 - e^{-z}$ and the fact that R_n is non-decreasing with respect to the t variable, we have that there is $K > 0$ such that

$$|\rho_n^2(t,x)\mathbb{1}_{|x|\geq\tilde{c}t}| \leq K R_n(t,x)\mathbb{1}_{|x|\geq\tilde{c}t} \leq K R_n\left(\frac{|x|}{\tilde{c}}, x\right).$$

We are in the same situation than in the previous step, we can use the exponential estimates from Lemma 3.9 to find that there is $q > 0$ such that

$$|\rho_n^2(t,x)\mathbb{1}_{|x|\geq\tilde{c}t}| \leq K e^{-q|x|}.$$

On the other hand, we have

$$|\rho_n^2(t,x)\mathbb{1}_{|x|<\tilde{c}t}| \leq \left|S_{n-1}^\star - R_{n-1}\left(\tfrac{|x|}{\tilde{c}}\wedge t, x\right)\right|\mathbb{1}_{|x|<\tilde{c}t} \leq \left|S_{n-1}^\star - R_{n-1}\left(\frac{|x|}{\tilde{c}}, x\right)\right|.$$

Because $\mathcal{H}_{n-1}$ holds true and because $\tilde{c} < c_{n-1}$, the quantity $x \mapsto |S_{n-1}^\star - R_{n-1}(\frac{|x|}{\tilde{c}}, x)|$ goes to zero as $|x|$ goes to $+\infty$.

Therefore, we have proven that $\rho_n^1(t,x) + \rho_n^2(t,x) \leq \varepsilon(x)$, where $\varepsilon(x) \underset{|x|\to+\infty}{\longrightarrow} 0$, the result follows. □

Combining all that precedes, we are now in position to prove Proposition 3.7.

Proof of Proposition 3.7. If $n = 1$, then we already have the result as explained in Remark 3.2. We suppose from now on that $N^\star \geq 2$ and $n \geq 2$.

Assume that $\mathcal{H}_{n-1}$ holds true. Then, owing to Remark 3.6, we have $R_{n-1} \leq S_{n-1}^\star + \varepsilon_n(x)$, for some function ε_n such that $\varepsilon_n(x) \underset{|x|\to+\infty}{\longrightarrow} 0$. Combining this with Lemma 3.4 yields that

$$\partial_t R_n \leq d_n \Delta R_n + \mu_n S_{n-1}^\star\Big(1 - e^{-\frac{\alpha_n}{\mu_n}R_n}\Big) - \mu_n R_n + \mu_n \varepsilon_n + \mu_n I_n^0.$$

On the other hand, Lemma 3.11 yields that there is $\tilde{\varepsilon}_n$ such that $\tilde{\varepsilon}_n(x) \underset{|x|\to+\infty}{\longrightarrow} 0$ so that

$$\partial_t R_n \geq d_n \Delta R_n + f_n(R_n) - \tilde{\varepsilon}_n. \tag{3.26}$$

Therefore, up to renaming $\max\{|\tilde{\varepsilon}_n|, \mu_n|\varepsilon_n| + \mu_n I_n^0\}$ as ε_n, the result follows. □

3.3 Propagation of R_n for $n \in [\![1, N^\star]\!]$

The aim of this section is to prove Proposition 3.1. We will proceed by induction: assuming $\mathcal{H}_{n-1}$, we prove that $\mathcal{H}_n$ holds true.

As a first step, we prove (3.10a) in a weaker form, in the sense that we show the convergence without the speed. This is outlined in the following lemma.

Lemma 3.12 *Let $n \in [\![1, N^\star]\!]$ and assume that $\mathcal{H}_{n-1}$ holds true. Then*

$$R_n(t,x) \underset{t\to+\infty}{\longrightarrow} R_n^\infty(x), \tag{3.27}$$

locally uniformly. Moreover

$$R_n^\infty(x) \underset{|x|\to+\infty}{\longrightarrow} S_n^\star. \tag{3.28}$$

Observe that Lemma 3.12 differs from Remark 3.6, since we claim (3.27)-(3.28) under $\mathcal{H}_{n-1}$, and not $\mathcal{H}_n$.

We also recall that $\mathcal{H}_0$ is vacuously true, that is, when $n = 1$, there is no hypothesis in the lemma, and we already know that the result is true, as explained in Remark 3.2.

Proof of Lemma 3.12. Assume that $\mathcal{H}_{n-1}$ holds true.

- *Step 1. R_n converges locally uniformly.* Because the function $R_n(t,x)$ is non-decreasing with respect to t, and because it is uniformly bounded — thanks to Lemma 3.9 —, there is

$R_\infty(x)$ such that $R_n(t,x) \underset{t\to+\infty}{\longrightarrow} R_n^\infty(x)$. This convergence is pointwise. However, $R_n(t,x)$ solves (3.7), therefore, owing to standard parabolic regularity theory [33], the convergence is actually locally uniform, and moreover the limit function $R_n^\infty(x)$ solves

$$-\varepsilon_n \le -d_n\Delta R_n^\infty - f_n(R_n^\infty) - \mu_n I_n^0 \le \varepsilon_n.$$

In the rest of the proof, we take $\rho \in \mathbb{R}$ to be an arbitrary limit point of $R_n^\infty(x)$ when $|x|$ goes to $+\infty$. This means that there is a sequence $(x_k)_{k\in\mathbb{N}}$ such that $|x_k| \to +\infty$ and

$$\rho = \lim_{k\to+\infty} R_n^\infty(x_k).$$

The rest of the proof consists in showing that $\rho = S_n^\star$.

• *Step 2. Either $\rho = 0$ or $\rho = S_n^\star$.* We denote

$$\rho_k(x) := R_n^\infty(x + x_k).$$

It is classical from elliptic regularity theory [22] that, up to extraction,

$$\rho_k(x) \underset{k\to+\infty}{\longrightarrow} \rho_\infty(x),$$

where ρ_∞ solves the elliptic equation

$$-d_n\Delta\rho_\infty - f_n(\rho_\infty) = 0.$$

Moreover, because R_n is uniformly bounded, the same holds true for ρ_∞. It is a classical result in reaction-diffusion equations theory that the only bounded solutions of this equation are the zeros of the function f_n, that are the constants 0 and $S_n^\star$ (see [1] for instance).

By definition of ρ_k, we have

$$\rho = \lim_{k\to+\infty} \rho_k(0),$$

therefore, either $\rho = 0$ or $\rho = S_n^\star$.

• *Step 3. Proof that $\rho > 0$.* Assume by contradiction that $\rho = 0$. Arguing as in the previous step, this implies that $R_n^\infty(\cdot + x_k) \to 0$ locally uniformly as $k \to +\infty$.

Now, let us show that there are $k \in \mathbb{N}$ and $T > 0$ large enough so that

$$\partial_t I_n \ge d_n\Delta I_n + (\alpha_n(S_{n-1}^\star - \varepsilon)e^{-\frac{\alpha_n}{\mu_n}\varepsilon} - \mu_n)I_n, \qquad \forall t > T,\ \forall x \in B_R(-x_k). \tag{3.29}$$

Because $R_n^\infty(\cdot + x_k) \to 0$ locally uniformly as k goes to $+\infty$, we can find k large enough so that

$$R_n^\infty(x + x_k) \le \varepsilon, \qquad \forall x \in B_R(0).$$

Therefore, thanks to (3.3), we have, for any $t > 0$ and $x \in B_R(-x_k)$,

$$S_{n-1}(t,x) \ge R_{n-1}(t,x)e^{-\frac{\alpha_n}{\mu_n}R_n(t,x)} \ge R_{n-1}(t,x)e^{-\frac{\alpha_n}{\mu_n}R_n^\infty(x)} \ge R_{n-1}(t,x)e^{-\frac{\alpha_n}{\mu_n}\varepsilon}.$$

Hence, because $\mathcal{H}_{n-1}$ holds true, up to increasing k if needed, and choosing $T > 0$ large enough, we have, for all $t > T$ and $x \in B_R(-x_k)$,

$$S_{n-1}(t,x) \ge (S_{n-1}^\star - \varepsilon)e^{-\frac{\alpha_n}{\mu_n}\varepsilon}.$$

This latter inequality implies that (3.29) holds true.

Let now λ be the principal eigenvalue of the operator $-d_n\Delta - (\alpha_n(S_n^\star - \varepsilon)e^{-\frac{\alpha_n}{\mu_n}\varepsilon} - \mu_n)$ on $B_R(-x_k)$ with Dirichlet boundary conditions, and let ϕ be a positive eigenfunction associated to the principal eigenvalue (its existence is guaranteed by the Krein-Rutman theorem [31]. Therefore, $\phi \in C^2(B_R(-x_k))$ is such that $\phi > 0$ on $B_R(-x_k)$, $\phi = 0$ on $\partial B_R(-x_k)$ and

$$-d_n\Delta\phi - (\alpha_n(S_n^\star - \varepsilon)e^{-\frac{\alpha_n}{\mu_n}\varepsilon} - \mu_n)\phi = \lambda\phi.$$

Let $v(t,x) = \phi(x)e^{-\lambda t}$, it solves

$$\partial_t v = d_n\Delta v + (\alpha_n(S_{n-1}^\star - \varepsilon)e^{-\frac{\alpha_n}{\mu_n}\varepsilon} - \mu_n)v, \qquad t > T,\ x \in B_R(-x_k)$$

with Dirichlet boundary condition $v(t,x) = 0$ for $x \in \partial B_R(-x_k)$ and $t > T$, and up to multiplying ϕ by a small constant, we can ensure that $v(T,\cdot) \leq I_n(T,\cdot)$. It then follows from the parabolic comparison principle that $I_n(t,x) \geq v(t,x)$ for all $t > T$ and $x \in B_R(-x_k)$. It is classical (see [6]) that, up to taking R large enough, the principal eigenvalue λ can be made as close as we want to $-(\alpha_n(S_n^\star - \varepsilon)e^{-\frac{\alpha_n}{\mu_n}\varepsilon} - \mu_n)$, which is strictly negative (because $n \leq N^\star$).

Therefore, $v(t,x) = \phi(x)e^{-\lambda t} \to +\infty$ for all x in $B_R(-x_k)$ as $t \to +\infty$, which is in contradiction with the boundedness of I_n given by Lemma 3.5.

In conclusion, we have proved that $\rho = S_n^\star$, that is, $R_n^\infty(x)$ indeed converges toward $S_n^\star$ as $|x|$ goes to $+\infty$. □

We now turn to the proof of Proposition 3.1. The idea is to compare R_n with a function solution of a *bistable* equation. Bistable reaction-diffusion equations are PDE of the form

$$\partial_t u = d\Delta u + f(u),$$

where the function f vanishes at three points: $0 < \theta < \sigma$ and is such that $f < 0$ on $(0,\theta)$ and $f > 0$ on (σ,θ). There is a wide literature on this specific type of equations [1, 3, 16, 38]. In particular, we recall the two following technical results that we shall need:

Proposition 3.13 (Sufficient condition for spreading in bistable equations) *Let f be a Lipschitz continuous function such that there are $0 < \theta < \sigma$ such that $f(0) = f(\theta) = f(\sigma) = 0$ and $f < 0$ on $(0,\theta)$ and $f > 0$ on (θ,σ). Assume that $\int_0^\sigma f(x)dx > 0$.*

Let u be the solution to

$$\partial_t u = d\Delta u + f(u), \qquad t > 0,\ x > 0, \tag{3.30}$$

with Dirichlet boundary condition $u(t,0) = 0$ for $t > 0$ and with initial datum $u_0 \geq 0$ compactly supported.

Then, there is $c^\star > 0$ such that, for all $\varepsilon > 0$, there is $L > 0$ such that if $u_0 \geq (\theta + \varepsilon)\mathbb{1}_{[0,L]}$, the function u spreads toward σ with speed $c^\star > 0$ (which depends only on f) in the sense that

$$\sup_{\delta < x < ct} |u(t,x) - \sigma| \underset{\delta, t\to+\infty}{\longrightarrow} 0, \qquad \forall c \in (0, c^\star),$$

and

$$\sup_{x > ct} |u(t,x)| \underset{t\to+\infty}{\longrightarrow} 0, \qquad \forall c > c^\star.$$

This result is classical and is based on constructing appropriate compactly supported subsolutions to (3.30). We refer to [1, 16] for proofs of this facts.

The next proposition tells us that, when we have a family of bistable nonlinearities that converge to a KPP nonlinearity, then the spreading speeds associated also converge. We shall use it with our KPP nonlinearities f_n defined in (3.12).

Proposition 3.14 (Convergence of bistable speeds) *Let $n \in [\![1, N^\star]\!]$ and consider the KPP nonlinearity f_n defined in* (3.12)*.*

Let $(f_\eta)_{\eta>0}$ be a family of bistable nonlinearities such that $f_\eta \to f_n$ locally uniformly as $\eta \to 0$. For each $\eta > 0$, let us denote c_η the corresponding speed of spreading given by Proposition 3.13. Then

$$c_\eta \underset{\eta \to 0}{\longrightarrow} c_n.$$

Unlike the speed of propagation for KPP reaction-diffusion equations, there is no explicit formula for the speed of propagation for bistable equations. There are variational formulas (see [24] for instance), but the proof of Proposition 3.14 can be obtained directly by taking the limit of the bistable traveling fronts for each f_η, see [38, Proposition 2.6] (we also refer to [3] for a different approach).

We are now in position to prove Proposition 3.1, which establishes the propagation of R_n.

Proof of Proposition 3.1. The proof is done by induction. Let $n \in [\![1, N^\star]\!]$ and assume that $\mathcal{H}_{n-1}$ holds true. Let us show that $\mathcal{H}_n$ also holds true.

Let $\eta, L > 0$ to be chosen small enough and large enough after.

Because we assume that $\mathcal{H}_{n-1}$ holds true, Proposition 3.7 tells us that R_n solves (3.13) (the perturbed KPP equation), where $\varepsilon_n(x)$ goes to zero as $|x|$ goes to $+\infty$.

Therefore, we can take $R > 0$ such that $|\varepsilon_n(x)| \leq \eta$ for $x > R$, so that the function R_n satisfies

$$\partial_t R_n \geq d_n \Delta R_n + f_n(R_n) - \eta, \qquad t > 0,\, x > R. \tag{3.31}$$

We now prove that the function R_n spreads toward $S_n^\star$ with the wanted speed toward the right (when $x \to +\infty$), the spreading toward the left (for $x \to -\infty$) can be done similarly.

Owing to Lemma 3.12, up to increasing R if needed, we can find $T > 0$ large enough so that

$$R_n(t,x) \geq S_n^\star - \eta, \qquad \forall t \geq T,\, \forall x \in [R, R+L].$$

We define $\rho(t,x) := R_n(t,x) + \eta$. It satisfies the three following properties

$$\begin{cases} \rho(t,x) \geq \eta, & \forall t > T,\, \forall x > R, \\ \rho(T,x) \geq S_n^\star, & \forall x \in [R, R+L], \\ \partial_t \rho \geq d_n \Delta \rho + f_\eta(\rho), & t > T,\, x > R, \end{cases}$$

where (see Figure XII below),

$$f_\eta(v) := \begin{cases} f_n(v-\eta) - \eta, & \text{if} \quad v \geq \eta, \\ -v, & \text{if} \quad v \in (0, \eta). \end{cases}$$

Observe that in the last inequality we can use f_η as defined and not simply $f_n(\cdot - \eta) - \eta$, because the function ρ is always larger that η, therefore, the values taken by $f_\eta(v)$ for $v < \eta$ do not matter.

The function f_η is a bistable nonlinearity in the sense defined above, provided η is sufficiently small. The function f_η vanishes at $x = 0$ and at two other points $0 < \theta < \sigma$. Observe that, as η goes to 0^+, we have $\theta \to 0$ and $\sigma \to S_n^\star$. The graph of f_η is depicted below.

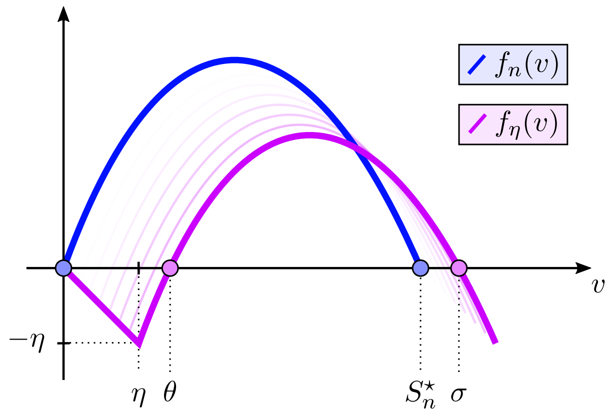

Figure XII — *Representation of the functions f_n and f_η.*

We take η small enough so that $\theta < S_n^\star$ and we denote v the solution to the following equation

$$\partial_t v = d_n \Delta v + f_\eta(v), \qquad t > T,\ x > R, \tag{3.32}$$

with "initial" datum $v(T, x) = S_n^\star \mathbb{1}_{[R,R+L]}(x)$ and Dirichlet boundary condition $v(t, R) = 0$ for all $t > T$.

Thanks to Proposition 3.13, we can choose L large enough so that the solution v of (3.32) spreads toward σ with some speed that we denote c_η (to emphasize the dependence on η).

Let us take $c \in (0, c_n)$. Up to taking η small enough, owing to Proposition 3.14, we can ensure $c_\eta > c$.

The parabolic comparison principle then implies that $\rho \geq v$ for any $t > T$ and $x > R$. Therefore, because $\rho = R_n + \eta$ and because R_n is non-decreasing with respect to t, we find

$$v(t, x) - \eta \leq R_n(t, x) \leq R_n^\infty(x), \qquad \forall t > T,\ x > R,$$

hence, for all $t > T$,

$$\sup_{\delta<|x|<ct} |R_n(t, x) - S_n^\star| \leq \max\left\{ \sup_{\delta<x<ct} |R_n^\infty(x) - S_n^\star|, \sup_{\delta<x<ct} |v(t, x) - \sigma| + |\sigma - S_n^\star - \eta| \right\}.$$

Because v spreads toward σ with speed $c_\eta > c$ and because $R_n^\infty(x)$ goes to $S_n^\star$ as x goes to $+\infty$, we find that

$$\limsup_{\delta, t \to +\infty} \sup_{\delta<|x|<ct} |R_n(t, x) - S_n^\star| \leq |\sigma - S_n^\star - \eta|.$$

We have $\sigma \to S_n^\star$ as $\eta \to 0$. Because η is arbitrary, we find

$$\limsup_{\delta, t \to +\infty} \sup_{\delta<|x|<ct} |R_n(t, x) - S_n^\star| = 0,$$

and this is true for all $c \in (0, c_n)$, that is, R_n spreads at least with speed c_n toward $S_n^\star$.

In addition, R_n converges toward $S_n^\star$ at most with speed c_n. Indeed, it follows from Lemma 3.9 (specifically (3.15)) that

$$\sup_{|x|>ct} |R_n(t,x)| \underset{t\to+\infty}{\longrightarrow} 0, \qquad \forall c > c_n.$$

Therefore, we have proven that, if $n \in [\![0, N^\star - 1]\!]$ and if $\mathcal{H}_{n-1}$ is true, then $\mathcal{H}_n$ also holds true. Proposition 3.1 is proved by induction. $\square$

3.4 Proof of Theorem 2.4

We are now in position to prove Theorem 2.4: for $n \in [\![1, N^\star]\!]$, the opinion n spreads while for $n > N^\star$, the opinion disappears. We start with a technical lemma.

Lemma 3.15 *Assume $N^\star < +\infty$. For all $n \geq N^\star + 1$, there is $\varepsilon \in L^\infty(\mathbb{R})$ such that $\varepsilon(x) \underset{|x|\to+\infty}{\longrightarrow} 0$ and $R_n \leq \varepsilon$.*

Proof of Lemma 3.15.

• *Step 1. The case $n = N^\star + 1$.*

Let us start with proving that there is $\varepsilon \in L^\infty(\mathbb{R})$, with $\varepsilon(x) \underset{|x|\to+\infty}{\longrightarrow} 0$, such that $R_{N^\star+1} \leq \varepsilon$. We write n instead of $N^\star + 1$ for the sake of readability. Owing to Lemma 3.4, we have

$$\partial_t R_n \leq d_n \Delta R_n + \mu_n R_{n-1}\Big(1 - e^{-\frac{\alpha_n}{\mu_n} R_n}\Big) - \mu_n R_n + \mu_n I_n^0, \qquad t > 0,\, x \in \mathbb{R}.$$

Because $\mathcal{H}_{N^\star}$ holds true, owing to Remark 3.6, we have that there is $\tilde\varepsilon \in L^\infty(\mathbb{R})$ such that $\tilde\varepsilon(x) \underset{|x|\to+\infty}{\longrightarrow} 0$, and

$$\partial_t R_n \leq d_n \Delta R_n + \mu_n S_{N^\star}^\star\Big(1 - e^{-\frac{\alpha_n}{\mu_n} R_n}\Big) - \mu_n R_n + \tilde\varepsilon, \qquad t > 0,\, x \in \mathbb{R}.$$

Owing to Lemma 3.5, we also know that $R_n \leq K_n$ for some $K_n > 0$.

Let $u(t,x)$ be solution of

$$\partial_t u = d_n \Delta u + f_n(u) + \tilde\varepsilon, \qquad t > 0,\, x \in \mathbb{R}, \tag{3.33}$$

with initial datum $u(0,\cdot) = K_n$. Up to increasing K_n if needed, we can ensure that K_n is a stationary supersolution of (3.33). Then, is it is classical that $u(t,x) \underset{t\to+\infty}{\searrow} U(x)$, where U is a stationary solution of (3.33). We have by comparison that $R_n(t,x) \leq R_n^\infty(x) \leq U(x)$ for all $t > 0$, $x \in \mathbb{R}$.

Let us show that $U(x) \underset{|x|\to+\infty}{\longrightarrow} 0$. To this aim, we take a sequence $(x_k)_{k\in\mathbb{N}}$ such that $|x_k| \to +\infty$ as $k \to +\infty$, and we define the translated functions $U_k(x) = U(x + x_k)$.

Owing to classical elliptic regularity results [22], we have that, up to extraction, $U_n \underset{n\to+\infty}{\longrightarrow} U_\infty$ (this convergences is locally in $W^{2,p}(\mathbb{R})$, for all $p > 1$), where U_∞ solves

$$d_n \Delta U_\infty + f_n(U_\infty) = 0,$$

and $0 \le U_\infty \le K_n$. The only function that satisfies this is the function everywhere equal to zero. Indeed, let $z(t)$ be solution of the ODE $\dot z = f_n(z)$ with initial datum $z(0) = K_n$. We have $z(t) \underset{t\to+\infty}{\longrightarrow} 0$.

The parabolic comparison principle implies that $z(t) \ge U_\infty(x)$ for all $t > 0$, and $x \in \mathbb{R}$. Taking the limit $t \to +\infty$ implies that $U_\infty \equiv 0$.

The result then holds true for $n = N^\star + 1$ with $\varepsilon = U$.

• *Step 2. The case $n > N^\star + 1$.*

We prove the result by induction: we show that, if there is $n \ge 2$ such that $R_{n-1} \le \varepsilon$, for some $\varepsilon \in L^\infty(\mathbb{R})$, with $\varepsilon(x) \underset{|x|\to+\infty}{\longrightarrow} 0$, then the same is true for R_n.

Owing to Lemma 3.4, we have that

$$\partial_t R_n \le d_n \Delta R_n + \mu_n R_{n-1}\Big(1 - e^{-\frac{\alpha_n}{\mu_n} R_n}\Big) - \mu_n R_n + \mu_n I_n^0.$$

Let $\eta > 0$ be such that $-\kappa := \alpha_n \eta - \mu_n < 0$. There is $R > 0$ such that

$$\partial_t R_n \le d_n \Delta R_n + \mu_n \eta\Big(1 - e^{-\frac{\alpha_n}{\mu_n} R_n}\Big) - \mu_n R_n + \mu_n I_n^0, \qquad \forall t > 0,\ \forall x > R.$$

Using the concavity of $z \mapsto \mu_n \eta\Big(1 - e^{-\frac{\alpha_n}{\mu_n} z}\Big) - \mu_n z$, we have

$$\partial_t R_n \le d_n \Delta R_n - \kappa R_n + \mu_n I_n^0, \qquad \forall t > 0,\ \forall x > R. \tag{3.34}$$

Owing to Lemma 3.5, there is $K_n > 0$ such that $R_n \le K_n$. Therefore, up to taking $A > 0$ large enough and $\lambda > 0$ small enough, the function

$$v(x) = A e^{-\lambda x}$$

is a stationary supersolution of the equation (3.34). By comparison, up to increasing A if needed, we have $R_n \le A e^{-\lambda x}$. Using the same arguments for $x < 0$, we find that $R_n \le A e^{-\lambda |x|}$.

Therefore, we have proven that, if there is $\varepsilon \in L^\infty(\mathbb{R})$, with $\varepsilon(x) \underset{|x|\to+\infty}{\longrightarrow} 0$, such that $R_{n-1} \le \varepsilon$, then the same holds true for R_n. The lemma follows by a direct induction. □

We now have all the tools to prove our main result, namely Theorem 2.4.

Proof of Theorem 2.4.

• *Proof of* (i)*: S_n spreads at most with speed c_n.* Let $n \in [\![1, N^\star]\!]$. Owing to Lemma 3.3, we have, $S_n \le R_n$. Therefore, Proposition 3.1 directly implies that

$$\sup_{|x|>(c_n+\varepsilon)t} |S_n(t,x)| \le \sup_{|x|>(c_n+\varepsilon)t} |R_n(t,x)| \underset{t\to+\infty}{\longrightarrow} 0.$$

This proves the point (i) of the theorem.

• *Proof of* (ii)*: S_n converges toward $S_n^\star$ in the intermediate region $c_{n+1}t < |x| < c_n t$.* For $n = 0$, this is a consequence of Remark 3.2. Let $n \in [\![1, N^\star]\!]$. We recall that we define $c_{N^\star+1} = 0$. Owing to Proposition 3.1, point (j), we have (when $n = N^\star$ this is actually a consequence of Lemma 3.15)

$$\sup_{|x|>(c_{n+1}+\varepsilon)t} |R_{n+1}| \underset{t\to+\infty}{\longrightarrow} 0,$$

and, owing to the point (jj), there holds

$$\sup_{(c_{n+1}+\varepsilon)t<|x|<(c_n-\varepsilon)t} |R_n - S_n^\star| \underset{t\to+\infty}{\longrightarrow} 0.$$

Therefore,

$$\sup_{(c_{n+1}+\varepsilon)t<|x|<(c_n-\varepsilon)t} |R_n e^{-\frac{\alpha_{n+1}}{\mu_{n+1}}R_{n+1}} - S_n^\star| \underset{t\to+\infty}{\longrightarrow} 0.$$

Owing to Lemma 3.3, we have that $R_n e^{-\frac{\alpha_{n+1}}{\mu_{n+1}}R_{n+1}} \leq S_n \leq R_n$, from which it follows that

$$\sup_{(c_{n+1}+\varepsilon)t<|x|<(c_n-\varepsilon)t} |S_n - S_n^\star| \underset{t\to+\infty}{\longrightarrow} 0.$$

This proves the point (ii) of the theorem.

• *Proof of* (iii)*: S_n converges toward $S_n^\dagger$ in the region $\delta < |x| < c_{n+1}t$.* Let $n \in [\![1, N^\star - 1]\!]$. Let $\varepsilon > 0$ be fixed. We want to prove that

$$\sup_{\delta<|x|<(c_{n+1}-\varepsilon)t} |S_n - S_n^\dagger| \underset{\delta,t\to+\infty}{\longrightarrow} 0.$$

First, owing to (3.9) from the proof of Lemma 3.4, we have for all $t > 0$ and $x \in \mathbb{R}$,

$$\partial_t R_{n+1} = d_{n+1}\Delta R_{n+1} + \mu_{n+1}(R_n - S_n) - \mu_{n+1}R_{n+1} + \mu_{n+1}I_{n+1}^0. \tag{3.35}$$

Now, let us chose three sequences $(t_k)_{k\in\mathbb{N}}, (x_k)_{k\in\mathbb{N}}, (\delta_k)_{k\in\mathbb{N}}$ such that $\delta_k, t_k \to +\infty$ and $\delta_k < |x_k| < (c_{n+1} - \varepsilon)t_n$.

We introduce the translated functions $R_{n+1}^k = R_{n+1}(\cdot + t_k, \cdot + x_k)$, $R_n^k = R_n(\cdot + t_k, \cdot + x_k)$ and $S_n^k = S_n(\cdot + t_k, \cdot + x_k)$.

For any $t > 0$ and $x \in \mathbb{R}$, we have, for k large enough,

$$|R_{n+1}(t + t_k, x + x_k) - S_{n+1}^\star| \leq \sup_{x+x_k\leq|y|\leq(c_{n+1}-\frac{\varepsilon}{2})t_k} |R_{n+1}(t + t_k, y) - S_{n+1}^\star|,$$

and, owing to Proposition 3.1, this goes to zero as k goes to $+\infty$.

Therefore, owing to parabolic regularity estimates, we have that R_{n+1}^k converges (up to a subsequence) locally uniformly in $W^{2,p}$, for all $p > 1$, to the function everywhere constant equal to $S_{n+1}^\star$. Similarly, R_n^k converges toward the function everywhere constant equal to $S_n^\star$.

Therefore, because we have

$$\partial_t R_{n+1}^k = d_{n+1}\Delta R_{n+1}^k + \mu_{n+1}(R_n^k - S_n^k) - \mu_{n+1}R_{n+1}^k + \mu_{n+1}I_{n+1}^0(\cdot + x_k),$$

taking the limit $k \to +\infty$ in this equation yields that, up to a subsequence,

$$\lim_{k\to+\infty} S_n^k(t, x) = S_n^\star - S_{n+1}^\star = S_n^\dagger,$$

and this convergence is locally uniform. Therefore, we have proven that, for each sequences $(x_k)_k$, $(\delta_k)_k$, $(t_k)_k$, as above, we have, up to a subsequence, $S_n(t_k, x_k) \underset{k\to+\infty}{\longrightarrow} S_n^\dagger$. The point (iii) of the theorem then holds true.

• *Proof of* (iv)*: S_n does not spread when $n > N^\star$.*

Now, owing to Lemma 3.15, we have that, for all $n \geq N^\star + 1$, there is $\varepsilon \in L^\infty(\mathbb{R})$, with $\varepsilon(x) \underset{|x|\to+\infty}{\longrightarrow} 0$, such that $R_n \leq \varepsilon$. Therefore, because $S_n \leq R_n$, the result holds true. □

4 Qualitative properties of the propagation sequences

This section is dedicated to the proof of Theorem 2.5 concerning the qualitative properties of $N^\star$, the maximal complexity obtained by the population.

The key point is to study the propagation sequences given by Definition 2.1. To do so, we can rephrase the definition of the propagation sequences as follows: let $S_0^\star$, $(d_n)_{n\in\mathbb{N}^\star}$, $(\alpha_n)_{n\in\mathbb{N}^\star}$, $(\mu_n)_{n\in\mathbb{N}^\star}$ be given. For $n \in \mathbb{N}$, define

$$\varphi_n(x) := x \Big/ \left(1 - e^{-\frac{\alpha_{n+1}}{\mu_{n+1}}x}\right).$$

This function is strictly increasing for $x > 0$, we have $\varphi_n(x) \underset{x\to 0}{\longrightarrow} \frac{\mu_{n+1}}{\alpha_{n+1}}$ and $\varphi_n(x) > x$ for all $x > 0$.

Let $N^\star$ and $(S_n^\star)_{n\in[\![0,N^\star]\!]}$ be the maximal complexity and the propagation sequence given by Definition 2.1. We have

$$S_n^\star = \varphi_n(S_{n+1}^\star), \qquad \forall n \in [\![0, N^\star - 1]\!], \tag{4.1}$$

and we can characterize $N^\star$ as follows:

$$N^\star \textit{ is the smallest integer } k \in \mathbb{N} \textit{ such that } \frac{\alpha_{k+1}}{\mu_{k+1}} S_k^\star \leq 1, \tag{4.2}$$

with the convention that $N^\star = +\infty$ if for all $k \in \mathbb{N}$, we have $\frac{\alpha_{k+1}}{\mu_{k+1}} S_k^\star > 1$.

Observe that, when all the coefficients $(\alpha_n)_{n\in\mathbb{N}}$ and $(\mu_n)_{n\in\mathbb{N}}$ are independent of n, we necessarily have $N^\star < +\infty$. Indeed, if this were not the case, then we could take the limit $n \to +\infty$ in (4.1): the function φ_n does not depend on n, and the sequence $(S_n^\star)_{n\in\mathbb{N}}$ is non-increasing, hence converges to a limit on $\mathbb{R}_+^\star$, and we would reach a contradiction because φ_n has no fixed point on $\mathbb{R}_+^\star$.

We start with proving the first point of Theorem 2.5, that is, we show that the maximal complexity is a non-decreasing function of the initial population $S_0^\star$ and of the transmission parameters $(\alpha_n)_{n\in\mathbb{N}^\star}$ and non-increasing with respect to the recovery parameters $(\mu_n)_{n\in\mathbb{N}^\star}$

Proof of Theorem 2.5 $\boxed{1}$. For $n \in \mathbb{N}^\star$, denote

$$\varphi_n(x) := x \Big/ \left(1 - e^{-\frac{\alpha_{n+1}}{\mu_{n+1}}x}\right) \qquad \text{and} \qquad \overline{\varphi_n}(x) := x \Big/ \left(1 - e^{-\frac{\overline{\alpha_{n+1}}}{\overline{\mu_{n+1}}}x}\right). \tag{4.3}$$

Owing to the hypotheses, we have $\varphi_n \leq \overline{\varphi_n}$.

We denote $N^\star := N^\star(S_0^\star, \alpha_n, \mu_n)$ and $\overline{N^\star} := N^\star(\overline{S_0^\star}, \overline{\alpha_n}, \overline{\mu_n})$ the maximal complexities reached by the systems with each set of parameters, and we denote $(S_n^\star)_{n\in[\![1,N^\star]\!]}$ and $\left(\overline{S_n^\star}\right)_{n\in[\![1,\overline{N^\star}]\!]}$ the corresponding propagation sequences.

Since $S_0^\star \leq \overline{S_0^\star}$, and because the functions φ_n and $\overline{\varphi_n}$ are non-decreasing and $\varphi_n \leq \overline{\varphi_n}$, it follows from (4.1) that, for each $n \in [\![1, \min\{N^\star, \overline{N^\star}\}]\!]$, we have

$$S_n^\star \leq \overline{S_n^\star},$$

and therefore

$$(\alpha_{n+1}/\mu_{n+1})S_n^\star \le (\overline{\alpha_{n+1}}/\overline{\mu_{n+1}})\overline{S_n^\star}.$$

Owing to (4.2), this gives

$$N^\star \le \overline{N^\star}.$$

□

We now turn to the second point of Theorem 2.5, that is we show that, if the parameters are chosen adequately, it is possible to have $N^\star = +\infty$. Moreover, it is possible to ensure that any proportion of the initial population can reach infinite complexity.

Proof of Theorem 2.5 $\boxed{2}$**.** Let us start by observing from (4.3), that, for all $n \in \mathbb{N}^\star$,

$$\varphi_n(x) \le x + \frac{1}{\lambda_{n+1}},$$

Where $\lambda_n := \frac{\alpha_n}{\mu_n}$. Therefore, owing to (4.1), we find, for all $n \in \mathbb{N}^\star$,

$$S_n^\star \ge S_0^\star - \sum_{k=1}^{n} \frac{1}{\lambda_k}.$$

Hence, up to choosing a sequence $(\lambda_n)_{n\in\mathbb{N}^\star}$ so that $\sum_{k=1}^{+\infty} \frac{1}{\lambda_k} \le \varepsilon$, we ensure that

$$\lim_{n\to+\infty} S_n^\star \ge S_0^\star - \varepsilon.$$

To conclude the proof, we need to apply Theorem 2.4, and to do so, we need the sequence of the speeds $(c_n)_{n\in\mathbb{N}^\star}$ to be decreasing (Assumption 2.2). One way to do this is to choose the diffusions $(d_n)_{n\in\mathbb{N}}$ so that this is true. We could also take all the diffusions equal (which is more natural from the modeling point of view) and multiply each α_n and μ_n by a coefficient $\varepsilon_n > 0$, so that the ratios λ_n are not changed while the speeds $c_n = 2\sqrt{d(\alpha_{n+1}\varepsilon_{n+1}S_n^\star - \varepsilon_{n+1}\mu_{n+1})}$ are decreasing. □

We now turn to the third point of Theorem 2.5, that is, we show that the maximal complexity $N^\star$ is equivalent to $e^{\alpha S_0^\star/\mu}/(\alpha S_0^\star/\mu) = e^{\mathscr{R}_0}/\mathscr{R}_0$.

Proof of Theorem 2.5 $\boxed{3}$**.** Let $S_0^\star > 0$ fixed. Let $(S_n^\star)_{n\in\mathbb{N}^\star}$ be the propagation sequence, as given in Definition 2.1.

First, observe that, because the sequence $(S_n^\star)_{n\in\mathbb{N}^\star}$ is decreasing, then the sequence of the speeds $(c_n)_{n\in\mathbb{N}^\star} = (2\sqrt{d(\alpha S_n^\star - \mu)})_{n\in\mathbb{N}^\star}$ is also strictly decreasing, hence Assumption 2.2 is verified. Therefore, Theorem 2.4 applies. We let $\lambda = \frac{\alpha}{\mu}$.

By denoting $\varphi(x) := x/(1-e^{-\lambda x})$, we have by definition

$$S_{n-1}^\star = \varphi(S_n^\star), \qquad \forall n \in [\![1, N^\star]\!].$$

Because the α_n and μ_n are independent of n, as explained in the beginning of this section, we have $N^\star < +\infty$. Owing to (4.2), we have that

$$S_{N^\star}^\star \le \frac{1}{\lambda} \qquad \text{and} \qquad S_{N^\star-1} > \frac{1}{\lambda}.$$

Therefore, because φ is increasing,

$$\frac{1}{\lambda} \le S_{N^\star-1}^\star \le \varphi(\frac{1}{\lambda}). \tag{4.4}$$

We denote $(u_n)_{n\in\mathbb{N}}$ the sequence defined by induction $u_{n+1} = \varphi(u_n)$ with $u_0 = \frac{1}{\lambda}$. Because $\varphi(x) > x$ the sequence $(u_n)_{n\in\mathbb{N}}$ is increasing. It follows that it diverges to $+\infty$.

Applying φ in in (4.4) $N^\star - 1$ times, we get

$$u_{N^\star-1} \le S_0^\star \le u_{N^\star}. \tag{4.5}$$

Now, define $\Phi(x) = \int_0^x \frac{e^{\lambda z}-1}{z} dz$. We have

$$|\Phi(u_{n+1}) - \Phi(u_n) - \Phi'(u_n)(u_{n+1} - u_n)| \le \frac{1}{2} \sup_{z\in[u_n,u_{n+1}]} |\Phi''(z)|(u_{n+1} - u_n)^2. \tag{4.6}$$

We have $u_{n+1} - u_n = \frac{u_n}{e^{\lambda u_n}-1} = \frac{1}{\Phi'(u_n)}$, and this goes to zero when n goes to $+\infty$.

On the other hand, it is clear that Φ'' is increasing for $z > 0$. Therefore,

$$\frac{1}{2} \sup_{z\in[u_n,u_{n+1}]} |\Phi''(z)|(u_{n+1} - u_n)^2 \le \frac{\Phi''(u_{n+1})}{2\Phi'(u_n)^2} \le \lambda \frac{u_n^2}{u_{n+1}} \frac{e^{\lambda u_{n+1}}}{(e^{\lambda u_n} - 1)^2},$$

and because $u_{n+1} - u_n$ goes to zero when n goes to $+\infty$, we have that the right-hand side in (4.6) goes to zero.

Therefore, $\Phi(u_{n+1}) - \Phi(u_n) \to 1$ when n goes to $+\infty$, and the Cesàro lemma implies

$$\frac{\Phi(u_n)}{n} \underset{n\to+\infty}{\longrightarrow} 1.$$

Now, applying Φ (which in increasing) to (4.5), we get

$$N^\star(S_0^\star) \underset{S_0^\star\to+\infty}{\sim} \Phi(S_0^\star).$$

By observing that $\Phi(x) \underset{x\to+\infty}{\sim} \frac{e^{\lambda x}}{\lambda x}$, the result follows. □

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