A Piston to Counteract Diffusion

The influence of an inward-shifting boundary on the heat equation in half-space

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Simulation for $\beta = 1$

Climate change, among other environmental factors, significantly impacts the distribution of biological populations. To better understand how these populations respond to dynamic external pressures, we propose a new diffusion model in a moving half-space, where the boundary evolves smoothly over time. By imposing a suitable boundary condition at the boundary, we prevent individuals from leaving the domain, so that the shifting boundary acts as an impermeable wall—a "piston"—that sweeps the individuals it encounters. This framework leads to an intricate interplay between the diffusion mechanism (which tends to spread the population) and the accumulation of individuals against the boundary.

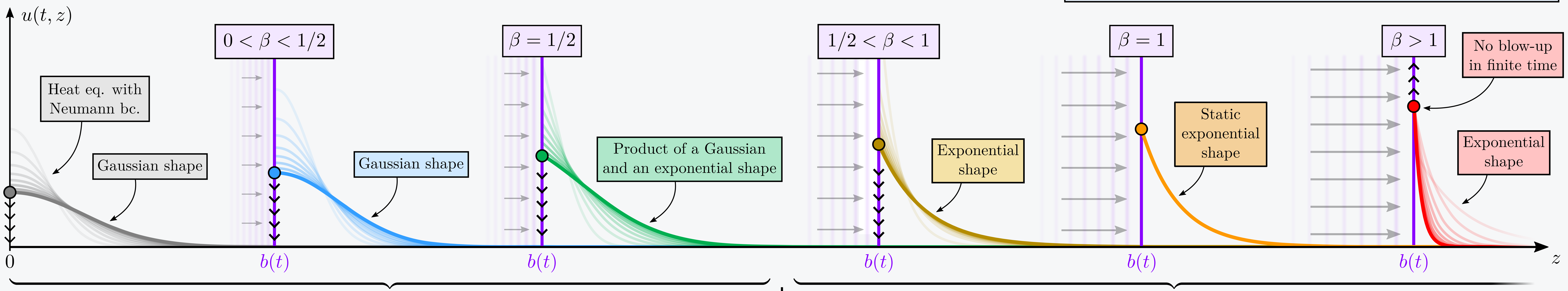
$$\begin{cases} \partial_t u = d\partial_{zz} u & t>0 \quad z > b(t) \\ -d\partial_z u = b'(t) u & t>0 \quad z = b(t) \end{cases}$$

$$v(t,x) := u(t, x + b(t))$$

$$\begin{cases} \partial_t v = d\partial_{xx} v + b'(t)\partial_x v & t>0 \quad x>0 \\ -d\partial_x v = b'(t) v & t>0 \quad x=0 \end{cases}$$

Guided by intuition stemming from microscopic motions, we focus here on algebraic growth of the boundary:

$$b(t) = c[(1+t)^\beta - 1]$$



Theorem (Critical and sub-critical cases). Assume $\beta \le 1/2$ and that the initial datum u_0 is bounded, nonnegative and compactly supported. Then

$$\lim_{t\to\infty} \left\| v(t,x) - \frac{1}{\sqrt{1+t}}\, W\left(\frac{x}{\sqrt{1+t}}\right) \right\|_{L^1_x(\mathbb{R}^*_+)} = 0.$$

Strategy of proof

(i) Self-similar rescaling $\longrightarrow$ $v(t,x) = \dfrac{1}{\sqrt{1+t}}\, w\Big(\overbrace{\log\sqrt{1+t}}^{=:\tau}, \overbrace{\frac{x}{\sqrt{1+t}}}^{=:y}\Big)$

Problem for w:

$$\begin{cases} \partial_\tau w = 2d\partial_{yy} w + \partial_y (y + \psi(\tau) w) & \tau > 0 \quad y > 0 \\ -2d\partial_y w = \psi(\tau) w & \tau > 0 \quad y = 0 \end{cases}$$

where $\begin{cases} \psi(\tau) \equiv c & \text{when } \beta = 1/2 \\ \psi(\tau) \xrightarrow{t\to\infty} 0 & \text{when } \beta \in (0, 1/2) \end{cases}$

Stationary solutions:

$$W(y) = \begin{cases} W(0)\exp\left(-\dfrac{y^2}{4d} - \dfrac{cy}{2d}\right) & \text{when } \beta = 1/2 \\ W(0)\exp\left(-\dfrac{y^2}{4d}\right) & \text{when } \beta \in (0,1/2) \end{cases}$$

(ii) Entropy methods to show the convergence.

Theorem (Super-critical cases). Assume $\beta > 1/2$ and that the initial datum u_0 is bounded, nonnegative and compactly supported. Then

$$\lim_{t\to\infty} \left\| v(t,x) - b'(t)\, W\big(b'(t)x\big) \right\|_{L^1_x(\mathbb{R}^*_+)} = 0.$$

Strategy of proof

(i) Self-similar rescaling $\longrightarrow$ $v(t,x) = b'(t)\, w\Big(\overbrace{\int_0^t [b'(s)^2]\,ds}^{=:\tau}, \overbrace{b'(t)x}^{=:y}\Big)$

Problem for w:

$$\begin{cases} \partial_\tau w = d\partial_{yy} w + \partial_y w - \frac{b''(t)}{(b'(t))^3}\partial_y(yw) & \tau > 0 \quad y > 0 \\ -d\partial_y w = w & \tau > 0 \quad y = 0 \end{cases}$$

where $\dfrac{b''(t)}{(b'(t))^3} \xrightarrow{t\to\infty} 0$

Stationary solutions: $W(y) = W(0)\exp\left(-\dfrac{y}{d}\right)$

(ii) When $\beta = 1$, everything is explicit. Direct estimates provide the convergence.

(iii) Treat $\frac{b''(t)}{(b'(t))^3}\partial_y(yw)$ as a vanishing source term, regarded as a perturbation around the case case $\beta = 1$ to apply Duhamel's principle.