The field-road diffusion model: fundamental solution and asymptotic behavior

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Biological motivation

There is increasing recognition that spatial spread can be favoured by so-called **fast diffusion channels**. For instance:

- in Canada, some GPS data revealed that **wolves travel faster along seismic lines** (i.e. narrow strips cleared for energy exploration), thus increasing their chances to meet a prey,
- **the spread of the black plague**, which killed about a third of the European population in the 14th century, was favoured by the trade routes – Silk Road.

To capture the phenomena induced by such fast diffusion channels, the so-called ***field-road* reaction diffusion system** $(\star)$ was proposed by Berestycki, Roquejoffre and Rossi in [1]. One considers a single population living on both domains

- $\mathbb{R}\times\mathbb{R}_+^*$ called the **field**,
- and $\mathbb{R}$ called the **road**.

We denote the population density by

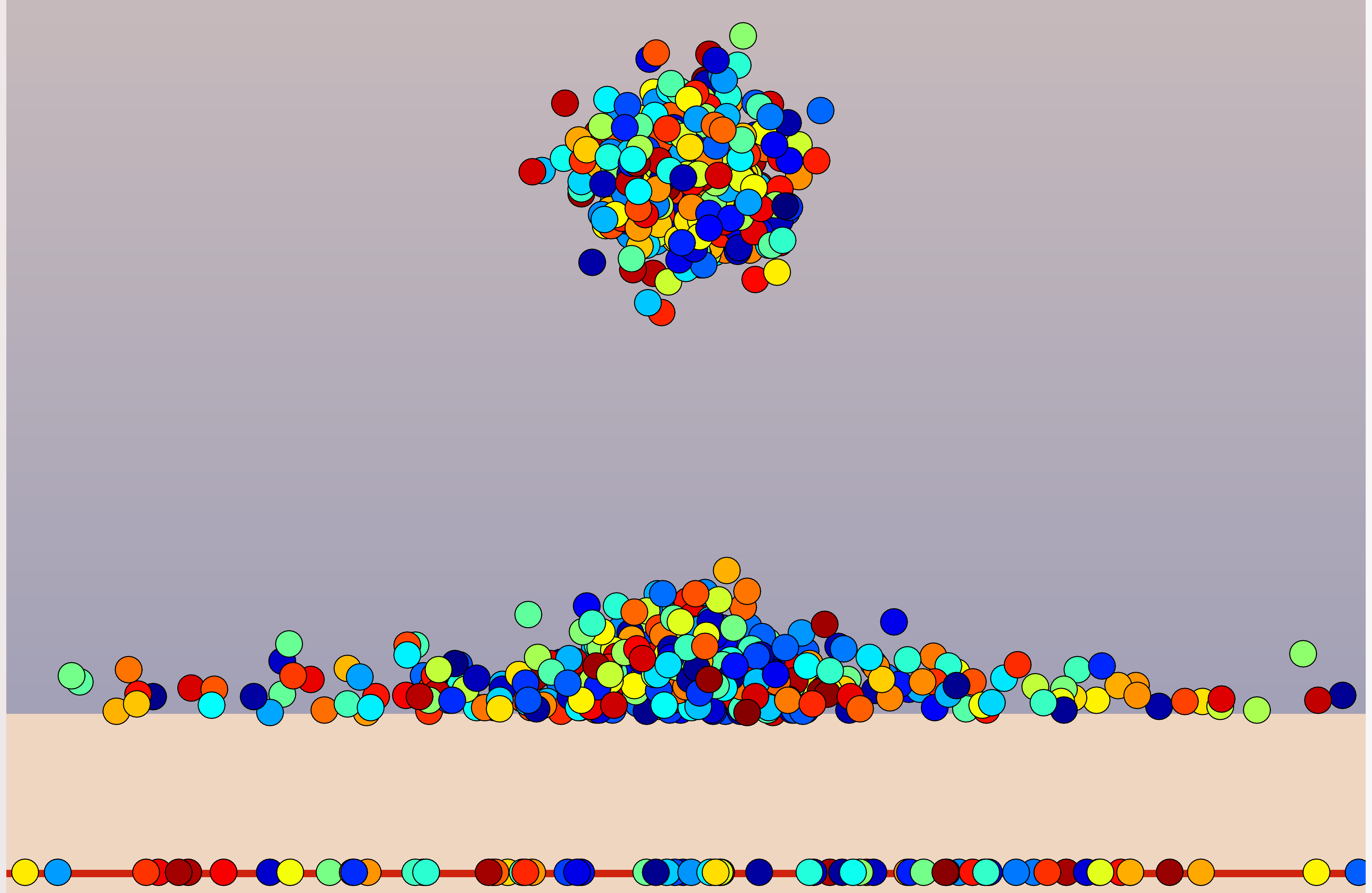
- $v(t,x,y)$ in the field,
- $u(t,x)$ in the road.

The individuals diffuse slower in the field than in the road ($d<D$) and the in-coming flux at the boundary of the field (2nd line of $(\star)$) exactly compensate the reaction term in the road PDE (3rd line of $(\star)$), so that **no individual is lost or created during the exchanges** between the two domains.

Reference

[1] H. Berestycki, J.-M. Roquejoffre, and L. Rossi, *The influence of a line with fast diffusion on Fisher-KPP propagation*, J. Math. Biol., (2013).

THE FIELD-ROAD SYSTEM
for fast diffusion channels



$$(\star)\begin{cases}\partial_t v = d\Delta v\\ -d\partial_y v|_{y=0} = \mu u - \nu v|_{y=0}\\ \partial_t u = D\partial_{xx} u + \nu v|_{y=0} - \mu u\end{cases}$$

Pure diffusive field-road system

It's movin' here!

Theorem (explicit solution) (2022)

The solution to problem $(\star)$ starting from the datum (v_0,u_0) assumed bounded is

$$v(t,x,y) = V(t,x,y) + \frac{\mu}{\sqrt{d}}\int_{\mathbb{R}}\Lambda(t,z,y)\,u_0(x-z)\,dz + \frac{\mu\,\nu}{\sqrt{d}}\int_0^t\int_{\mathbb{R}}\Lambda(s,z,y)\,V|_{y=0}(t-s,x-z)\,dz\,ds,$$

$$u(t,x) = e^{-\mu t}U(t,x) + \nu\int_0^t e^{-\mu(t-s)}\int_{\mathbb{R}} G(t-s,x-z)\,v|_{y=0}(s,z)\,dz\,ds,$$

where:

V and U are the solution to the Cauchy problems

$$\begin{cases}\partial_t V = d\Delta V,\\ \nu V|_{y=0} - d\partial_y V|_{y=0} = 0,\\ V|_{t=0} = v_0,\end{cases}\qquad \begin{cases}\partial_t U = D\partial_{xx}U,\\ U|_{t=0} = u_0,\end{cases}$$

$G = G(t,x)$ is the one-dimensional Heat kernel

and Λ is a rather technical function which manages migration from the road to the field.

In the statement below, the notation $B \lesssim B'$ means there is a constant $k = k(d,D,\mu,\nu) > 0$ such that $B \le kB'$.

Theorem (L^∞ decay rate) (2022)

Assume $D>d$, and let (v,u) the solution to problem $(\star)$ starting from the datum (v_0,u_0) assumed positive, bounded and compactly supported. Then

$$\|v(t,\cdot)\|_{L^\infty(\mathbb{R}^2_+)} \lesssim \frac{C_{v_0}\ln(1+t) + C_{v_0}^{u_0}}{1+t}, \qquad \forall t>0,$$

$$\|u(t,\cdot)\|_{L^\infty(\mathbb{R})} \lesssim \frac{C_{v_0}\ln(1+t) + C_{v_0}^{u_0}}{1+t}, \qquad \forall t>0,$$

for some nonnegative constant C_{v_0} depending on v_0, and some nonnegative constant $C_{v_0}^{u_0}$ depending on v_0 and u_0.

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