

THE FIELD-ROAD DIFFUSION MODEL:
FUNDAMENTAL SOLUTION AND
ASYMPTOTIC BEHAVIOR

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➤ BIOLOGICAL MOTIVATION

There is increasing recognition that spatial spread can be favoured by so-called **fast diffusion channels**. For instance:

- in Canada, some GPS data revealed that **wolves travel faster along seismic lines** (i.e. narrow strips cleared for energy exploration), thus increasing their chances to meet a prey,
- **the spread of the black plague**, which killed about a third of the European population in the 14th century, was favoured by the trade routes – Silk Road.

To capture the phenomena induced by such fast diffusion channels, the so-called **field-road reaction diffusion system** (★) was proposed by Berestycki, Roquejoffre and Rossi in [1]. One considers a single population living on both domains

- $\mathbb{R} \times \mathbb{R}_+^*$ called the **field**,
- and \mathbb{R} called the **road**.

We denote the population density by

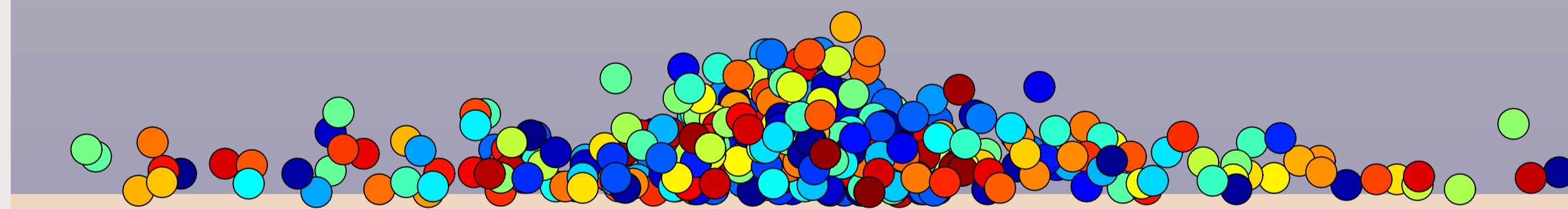
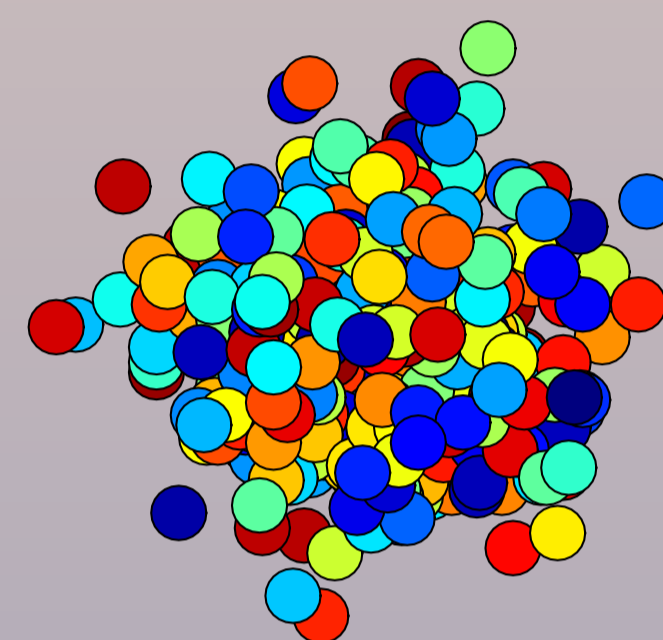
- $v(t, x, y)$ in the field,
- $u(t, x)$ in the road.

The individuals diffuse slower in the field than in the road ($d < D$) and the in-coming flux at the boundary of the field (2nd line of (★)) exactly compensates the reaction term in the road PDE (3rd line of (★)), so that **no individual is lost or created during the exchanges** between the two domains.

➤ REFERENCE

[1] H. BERESTYCKI, J.-M. ROQUEJOFFRE, AND L. ROSSI, *The influence of a line with fast diffusion on Fisher-KPP propagation*, J. Math. Biol., (2013).

THE FIELD-ROAD SYSTEM for fast diffusion channels



$$(\star) \begin{cases} \partial_t v = d\Delta v \\ -d\partial_y v|_{y=0} = \mu u - \nu v|_{y=0} \\ \partial_t u = D\partial_{xx} u + \nu v|_{y=0} - \mu u \end{cases}$$

Pure diffusive field-road system



It's movin' here!

THEOREM (EXPLICIT SOLUTION) (2022)

The solution to problem (★) starting from the datum (v_0, u_0) assumed bounded is

$$v(t, x, y) = V(t, x, y) + \frac{\mu}{\sqrt{d}} \int_{\mathbb{R}} \Lambda(t, z, y) u_0(x - z) dz + \frac{\mu \nu}{\sqrt{d}} \int_0^t \int_{\mathbb{R}} \Lambda(s, z, y) V|_{y=0}(t - s, x - z) dz ds,$$

$$u(t, x) = e^{-\mu t} U(t, x) + \nu \int_0^t e^{-\mu(t-s)} \int_{\mathbb{R}} G(t - s, x - z) v|_{y=0}(s, z) dz ds,$$

where:

V and U are the solution to the Cauchy problems

$$\begin{cases} \partial_t V = d\Delta V, \\ \nu V|_{y=0} - d\partial_y V|_{y=0} = 0, \\ V|_{t=0} = v_0, \end{cases} \quad \begin{cases} \partial_t U = D\partial_{xx} U, \\ U|_{t=0} = u_0, \end{cases}$$

$G = G(t, x)$ is the one-dimensional Heat kernel

and Λ is a rather technical function which manages migration from the road to the field.

In the statement below, the notation $B \lesssim B'$ means there is a constant $k = k(d, D, \mu, \nu) > 0$ such that $B \leq kB'$.

THEOREM (L^∞ DECAY RATE) (2022)

Assume $D > d$, and let (v, u) the solution to problem (★) starting from the datum (v_0, u_0) assumed positive, bounded and compactly supported. Then

$$\|v(t, \cdot)\|_{L^\infty(\mathbb{R}_+^2)} \lesssim \frac{C_{v_0} \ln(1+t) + C_{u_0}^{v_0}}{1+t}, \quad \forall t > 0,$$

$$\|u(t, \cdot)\|_{L^\infty(\mathbb{R})} \lesssim \frac{C_{v_0} \ln(1+t) + C_{u_0}^{u_0}}{1+t}, \quad \forall t > 0,$$

for some nonnegative constant C_{v_0} depending on v_0 , and some nonnegative constant $C_{u_0}^{v_0}$ depending on v_0 and u_0 .