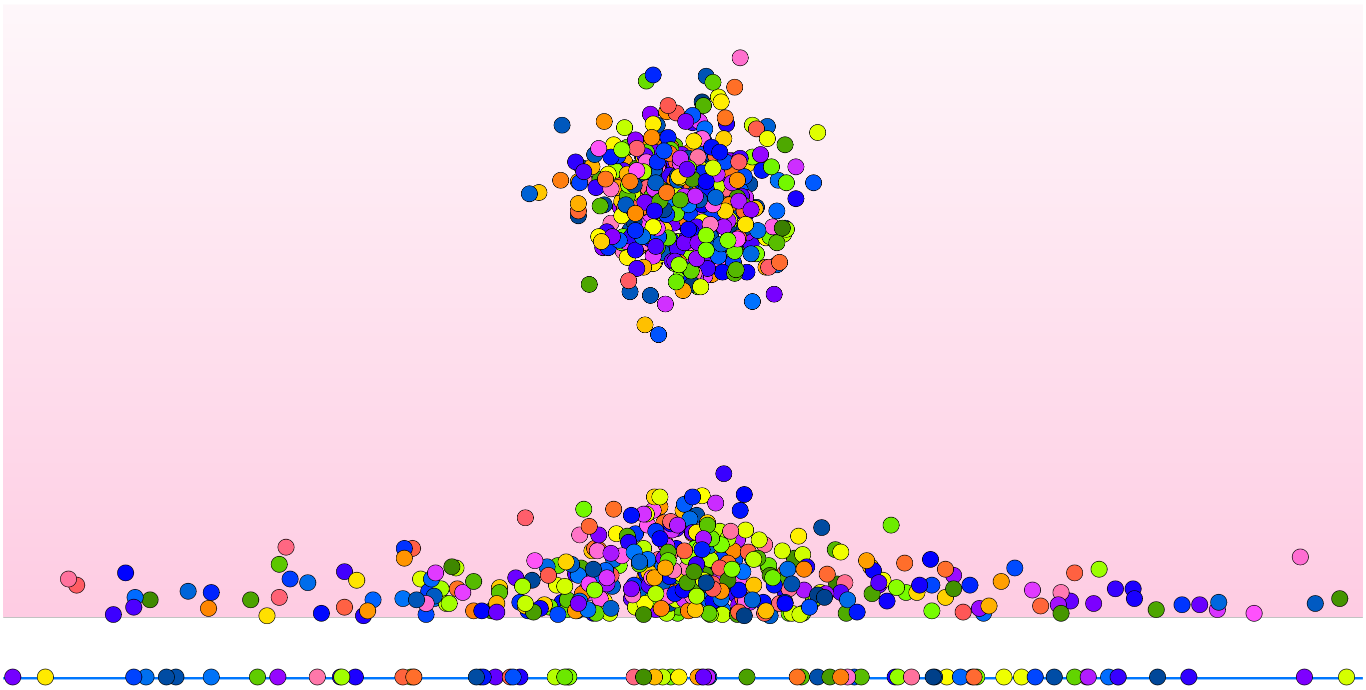
MatCha

(**MAT**hematical models for fast diffusion **CHA**nnels in population dynamics and epidemiology)

Annexe II



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SECTION I

Fujita on a Heat-exchanger-system

I.1 The linear model

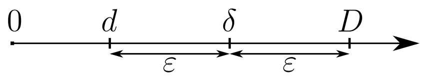
We consider in this section the following *Heat-exchanger-system*

$$\begin{cases} \partial_t u = d\Delta u - u + v, & t > 0, \quad x \in \mathbb{R} \\ \partial_t v = D\Delta v + u - v, & t > 0, \quad x \in \mathbb{R}. \end{cases} \tag{I.1}$$

Without loss of generality, one assumes that $D \geq d$ and one sets for convenience:

$$\delta := \frac{D+d}{2}, \qquad \text{and} \qquad \varepsilon := \frac{D-d}{2}.$$

Notice $d > 0$ requires $\delta > \varepsilon$.



Remark. If $\varepsilon = 0$, we have $d = D = \delta$; whence, calling $\sigma := u + v$ and $w := u - v$, one gets the uncoupled system

$$\begin{cases} \partial_t \sigma = \delta\Delta\sigma, & t > 0, \quad x \in \mathbb{R} \\ \partial_t w = \delta\Delta w - 2w & t > 0, \quad x \in \mathbb{R} \end{cases}$$

whose straight resolution yields

$$u(t,x) = \frac{1 + e^{-2t}}{2} \left[G_\delta(t, \bullet) * u_0\right](x) + \frac{1 - e^{-2t}}{2} \left[G_\delta(t, \bullet) * v_0\right](x)$$

$$v(t,x) = \frac{1 - e^{-2t}}{2} \left[G_\delta(t, \bullet) * u_0\right](x) + \frac{1 + e^{-2t}}{2} \left[G_\delta(t, \bullet) * v_0\right](x).$$

It is then clear that $\|u(t, \bullet)\|_\infty, \|v(t, \bullet)\|_\infty \lesssim t^{-1/2}$.

In a more general way the latter move does not work; however, we are able to provide an explicit expression of the Fourier transform of the solution:

$$\widehat{u}(t,\xi) = \frac{(\sqrt{s} + \varepsilon\xi^2)e^{-t(1-\sqrt{s})} + (\sqrt{s} - \varepsilon\xi^2)e^{-t(1+\sqrt{s})}}{2\sqrt{s}}\, e^{-\delta t\xi^2}\, \widehat{u}_0(\xi) + \frac{e^{-t(1-\sqrt{s})} - e^{-t(1+\sqrt{s})}}{2\sqrt{s}}\, e^{-\delta t\xi^2}\, \widehat{v}_0(\xi)$$

$$\widehat{v}(t,\xi) = \frac{e^{-t(1-\sqrt{s})} - e^{-t(1+\sqrt{s})}}{2\sqrt{s}}\, e^{-\delta t\xi^2}\, \widehat{u}_0(\xi) + \frac{(\sqrt{s} - \varepsilon\xi^2)e^{-t(1-\sqrt{s})} + (\sqrt{s} + \varepsilon\xi^2)e^{-t(1+\sqrt{s})}}{2\sqrt{s}}\, e^{-\delta t\xi^2}\, \widehat{v}_0(\xi)$$

with $\sqrt{s} := \sqrt{1+\varepsilon^2\xi^4}$. As in the latter remark, it seems that u and v can be split into a *persistent part* and a *residual part*:

$$u(t,x) = u_\infty(t,x) + u_r(t,x) \qquad \text{and} \qquad v(t,x) = v_\infty(t,x) + v_r(t,x),$$

with

$$\widehat{u}_\infty(t,\xi) = \frac{(\sqrt{s} + \varepsilon\xi^2)e^{-t(1-\sqrt{s})}}{2\sqrt{s}}\, e^{-\delta t\xi^2}\, \widehat{u}_0(\xi) + \frac{e^{-t(1-\sqrt{s})}}{2\sqrt{s}}\, e^{-\delta t\xi^2}\, \widehat{v}_0(\xi)$$

$$\widehat{u}_r(t,\xi) = \frac{(\sqrt{s} - \varepsilon\xi^2)e^{-t(1+\sqrt{s})}}{2\sqrt{s}}\, e^{-\delta t\xi^2}\, \widehat{u}_0(\xi) - \frac{e^{-t(1+\sqrt{s})}}{2\sqrt{s}}\, e^{-\delta t\xi^2}\, \widehat{v}_0(\xi)$$

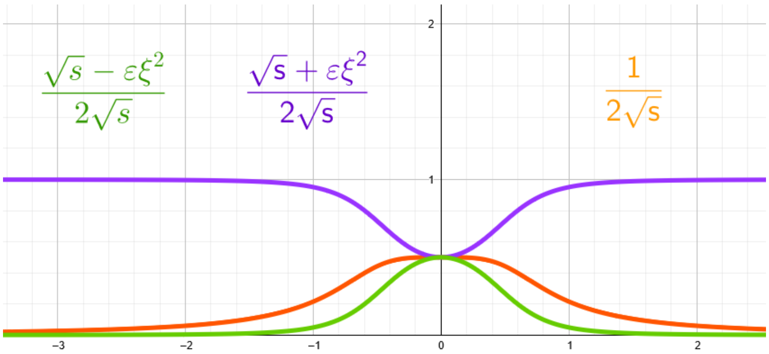
$$\widehat{v}_\infty(t,\xi) = \frac{e^{-t(1-\sqrt{s})}}{2\sqrt{s}}\, e^{-\delta t\xi^2}\, \widehat{u}_0(\xi) + \frac{(\sqrt{s} - \varepsilon\xi^2)e^{-t(1-\sqrt{s})}}{2\sqrt{s}}\, e^{-\delta t\xi^2}\, \widehat{v}_0(\xi)$$

$$\widehat{v}_r(t,\xi) = -\frac{e^{-t(1+\sqrt{s})}}{2\sqrt{s}}\, e^{-\delta t\xi^2}\, \widehat{u}_0(\xi) + \frac{(\sqrt{s} + \varepsilon\xi^2)e^{-t(1+\sqrt{s})}}{2\sqrt{s}}\, e^{-\delta t\xi^2}\, \widehat{v}_0(\xi).$$

Proposition 1 (The residual parts vanish exponentially fast)
Assuming u_0 and v_0 to be smooth, there is $c > 0$ such that $\|u_r(t,\bullet)\|_\infty < ce^{-2t}$ and $\|v_r(t,\bullet)\|_\infty < ce^{-2t}$. Furthermore, c is explicitly known and only depends on N and the L^1-norms of $\widehat{u}_0$ and $\widehat{v}_0$.

Lemma 2 (Boundedness)
One has,

$$\boxed{1}\quad 0 < \frac{\sqrt{s} - \varepsilon\xi^2}{2\sqrt{s}} \le \frac{1}{2} \le \frac{\sqrt{s} + \varepsilon\xi^2}{2\sqrt{s}} < 1, \qquad\qquad \boxed{2}\quad 0 < \frac{1}{2\sqrt{s}} \le \frac{1}{2},$$



Proof (Lemma 2)

We have $\sqrt{s} = \sqrt{1+\varepsilon^2\xi^4} \geq 1$, hence $\boxed{2}$ is clearly true; then for $\xi \neq 0$,

$$0 < \frac{\sqrt{s} - \varepsilon\xi^2}{2\sqrt{s}} = \frac{1}{2} - \frac{1}{2\sqrt{1+\frac{1}{\varepsilon^2\xi^4}}} < \frac{1}{2},$$

$$\frac{1}{2} < \frac{\sqrt{s} + \varepsilon\xi^2}{2\sqrt{s}} = \frac{1}{2} + \frac{1}{2\sqrt{1+\frac{1}{\varepsilon^2\xi^4}}} < 1,$$

and for $\xi = 0$, $\frac{\sqrt{s}\pm\varepsilon\xi^2}{2\sqrt{s}} = \frac{1}{2}$. That proves $\boxed{1}$. $\square$

Proof (Proposition 1)

It is sufficient to show the controls for the L^1-norm of $\widehat{u}(t,\bullet)$ and $\widehat{v}(t,\bullet)$, since

$$|u(t,x)| = \frac{1}{2\pi}\left|\int_{\mathbb{R}} \widehat{u}(t,\xi)e^{i\xi x}d\xi\right| \leq \frac{1}{2\pi}\int_{\mathbb{R}}\left|\widehat{u}(t,\xi)\right|d\xi = \frac{1}{2\pi}\left\|\widehat{u}(t,\bullet)\right\|_{L^1(\mathbb{R})}.$$

We have $1+\sqrt{s} \geq 2$, hence, using Lemma 2,

$$\int_{\mathbb{R}}\left|\widehat{u}_r(t,\xi)\right|d\xi \leq e^{-2t}\int_{\mathbb{R}}\frac{1}{2}\left|\widehat{u}_0(\xi)\right| + \frac{1}{2}\left|\widehat{v}_0(\xi)\right|d\xi$$

$$\int_{\mathbb{R}}\left|\widehat{v}_r(t,\xi)\right|d\xi \leq e^{-2t}\int_{\mathbb{R}}\frac{1}{2}\left|\widehat{u}_0(\xi)\right| + \left|\widehat{v}_0(\xi)\right|d\xi,$$

whence

$$\|u_r(t,\bullet)\|_\infty \leq e^{-2t}\left(\frac{1}{4\pi}\left\|\widehat{u}_0\right\|_{L^1(\mathbb{R})} + \frac{1}{4\pi}\left\|\widehat{v}_0\right\|_{L^1(\mathbb{R})}\right)$$

$$\|v_r(t,\bullet)\|_\infty \leq e^{-2t}\left(\frac{1}{4\pi}\left\|\widehat{u}_0\right\|_{L^1(\mathbb{R})} + \frac{1}{2\pi}\left\|\widehat{v}_0\right\|_{L^1(\mathbb{R})}\right). \quad \square$$

We are now investigating what kind of equation solve the persistent parts u_∞ and v_∞. On the Fourier-side, we own

$$\widehat{u}_\infty(t,\xi) = \left[\frac{\sqrt{s} + \varepsilon\xi^2}{2\sqrt{s}}\,\widehat{u}_0(\xi) + \frac{1}{2\sqrt{s}}\,\widehat{v}_0(\xi)\right] e^{t(\sqrt{s}-\delta\xi^2-1)} =: \widehat{u}_{0,\infty}(\xi)\, e^{t(\sqrt{s}-\delta\xi^2-1)}$$

$$\widehat{v}_\infty(t,\xi) = \left[\frac{1}{2\sqrt{s}}\,\widehat{u}_0(\xi) + \frac{\sqrt{s} - \varepsilon\xi^2}{2\sqrt{s}}\,\widehat{v}_0(\xi)\right] e^{t(\sqrt{s}-\delta\xi^2-1)} =: \widehat{v}_{0,\infty}(\xi)\, e^{t(\sqrt{s}-\delta\xi^2-1)}$$

hence $\widehat{u}_\infty(\bullet,\xi)$ and $\widehat{v}_\infty(\bullet,\xi)$ satisfy the Cauchy problems (ξ plays as a parameter)

$$\begin{cases} \partial_t \widehat{u}_\infty = (\sqrt{s} - \delta\xi^2 - 1)\,\widehat{u}_\infty \\ \widehat{u}_\infty|_{t=0} = \widehat{u}_{0,\infty} \end{cases} \qquad \begin{cases} \partial_t \widehat{v}_\infty = (\sqrt{s} - \delta\xi^2 - 1)\,\widehat{v}_\infty \\ \widehat{v}_\infty|_{t=0} = \widehat{v}_{0,\infty}. \end{cases}$$

As a consequence, by taking the inverse Fourier transform, we are led to these *non-local* and *uncoupled* diffusion equations:

$$\begin{cases} \partial_t u_\infty = \mathcal{F}_\xi^{-1}\left[\sqrt{1+\varepsilon^2\xi^4} - \delta\xi^2\right] * u_\infty - u_\infty, & t>0, \quad x \in \mathbb{R} \\ u_\infty|_{t=0} = u_{0,\infty} & x \in \mathbb{R} \end{cases}$$

$$\begin{cases} \partial_t v_\infty = \mathcal{F}_\xi^{-1}\left[\sqrt{1+\varepsilon^2\xi^4} - \delta\xi^2\right] * v_\infty - v_\infty, & t>0, \quad x \in \mathbb{R} \\ v_\infty|_{t=0} = v_{0,\infty} & x \in \mathbb{R}. \end{cases}$$

We set:

$$J(x) := \mathcal{F}_\xi^{-1}\left[\sqrt{1+\varepsilon^2\xi^4} - \delta\xi^2\right](x).$$

It is clear that J is a probability kernel since

$$\|J\|_{L^1(\mathbb{R})} = \left[\sqrt{1+\varepsilon^2\xi^4} - \delta\xi^2\right]\Big|_{\xi=0} = 1.$$

Now, near the origin,

$$\widehat{J}(\xi) = 1 - \delta\xi^2 + \frac{\varepsilon^2}{2}\xi^4 + o(\xi^4),$$

and, as $|\xi| \to \infty$,

$$\widehat{J}(\xi) = -d\xi^2 + \frac{1}{2\varepsilon\xi^2} + o\left(\frac{1}{\xi^2}\right).$$